

Idea solve systems of linear equations.

Solve $Ax=b$ for x

Here $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

Factorizations ... factor A into a product of easier to understand matrices

Algorithm

$$A = LU$$

↑ upper triangular
lower triangular

$$A = PLU$$

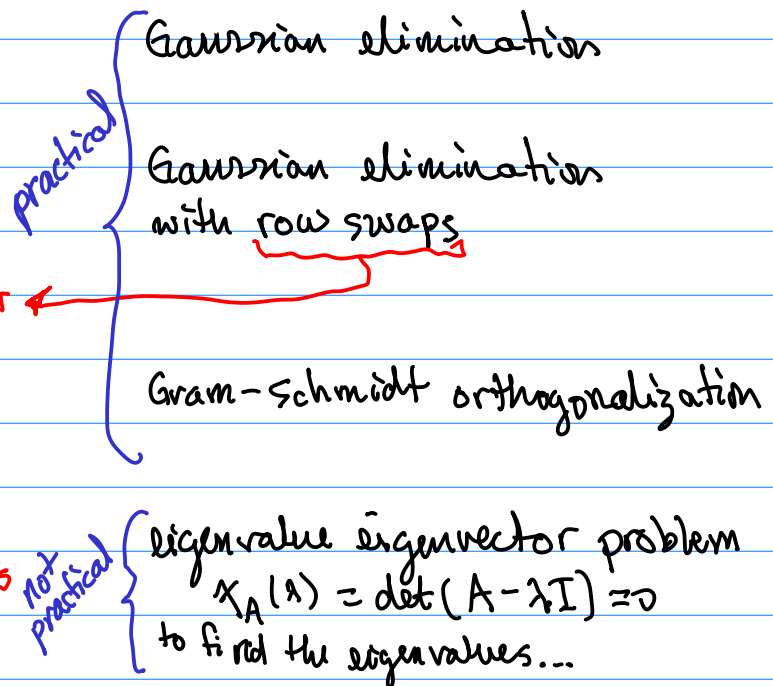
↑ upper triangular
lower triangular
permutation matrix $P^{-1} = P^T$

$$A = QR$$

↑ upper triangular
orthogonal $Q^{-1} = Q^T$

$$A = S^{-1}DS$$

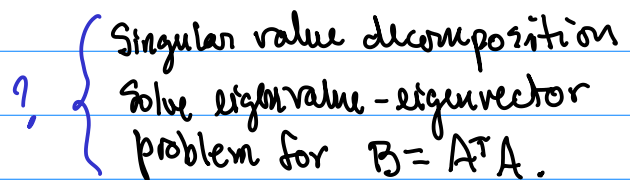
↑ matrix of eigenvectors
diagonal matrix



Iterative method of Francis.

$$A = U \Sigma V^T$$

↑ orthogonal
diagonal matrix
orthogonal matrix.



← eigenvectors of B normalized to unit length

Gaussian elimination

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$

do elimination steps to make a triangular matrix U .

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

elimination

$$r_2 \leftarrow r_2 - 4r_1$$

$$r_3 \leftarrow r_3 - 2r_1$$

magnifying the error if multipliers in the elimination are greater than one

Pivots

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -4 & -9 \end{bmatrix}$$

$$r_3 \leftarrow r_3 - \frac{4}{3}r_2$$

$$-9 - \frac{4}{3}(-6) = -1$$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & \frac{4}{3} & 1 \end{bmatrix}$$

all the entries in L were greater than one ...

Claim $A = LU$.

```
julia> U=[1 2 3; 0 -3 -6; 0 0 -1]
3x3 Matrix{Int64}:
 1  2  3
 0 -3 -6
 0  0 -1
```

```
julia> L=[1 0 0; 4 1 0; 2 4/3 1]
3x3 Matrix{Float64}:
 1.0  0.0  0.0
 4.0  1.0  0.0
 2.0  1.33333  1.0
```

```
julia> L*U
3x3 Matrix{Float64}:
 1.0  2.0  3.0
 4.0  5.0  6.0
 2.0  0.0 -3.0
```

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

Perform row swaps so the multipliers in the elimination steps are less than 1.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 0 & -3 \end{bmatrix}$$

$$r_1 \leftrightarrow r_2$$

$$r_2 \leftarrow r_2 - \frac{1}{4}r_1$$

$$r_3 \leftarrow r_3 - \frac{2}{4}r_1 = r_3 - \frac{1}{2}r_2$$

$$2 - \frac{1}{4}5 = \frac{8-5}{4} = \frac{3}{4}$$

$$3 - \frac{1}{4}6 = 3 - \frac{3}{2} = \frac{6-3}{2} = \frac{3}{2}$$

$$-3 - \frac{1}{2}6 = -3 - 3 = -6$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & 3/4 & 3/2 \\ 0 & -5/2 & -6 \end{bmatrix}$$

So the next elimination step also has a small mult.

$$r_2 \leftrightarrow r_3$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & -5/2 & -6 \\ 0 & 3/4 & 3/2 \end{bmatrix}$$

$$r_3 \leftarrow r_3 - \frac{(3/4)}{(-5/2)}r_2 = r_3 + \frac{3}{10}r_2$$

$$\frac{3}{2} + \frac{3}{10}(-6) = \frac{15-18}{10} = -\frac{3}{10}$$

$$U = \begin{bmatrix} 4 & 5 & 6 \\ 0 & -5/2 & -6 \\ 0 & 0 & -3/10 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{10} & 1 \end{bmatrix}$$

Careful here because $\frac{1}{2}$ and $\frac{1}{4}$ go in different places because of the row swap $r_2 \leftrightarrow r_3$ done later

```
julia> L=[1 0 0; 1/2 1 0; 1/4 -3/10 1]
3x3 Matrix{Float64}:
 1.0  0.0  0.0
 0.5  1.0  0.0
 0.25 -0.3  1.0
```

```
julia> U=[4 5 6; 0 -5/2 -6; 0 0 -3/10]
3x3 Matrix{Float64}:
 4.0  5.0  6.0
 0.0 -2.5 -6.0
 0.0  0.0 -0.3
```

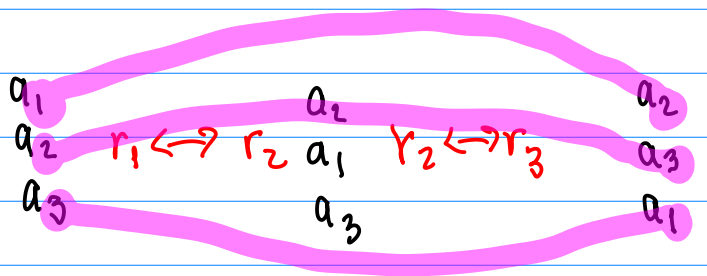
```
julia> L*U
3x3 Matrix{Float64}:
 4.0  5.0  6.0
 2.0  0.0 -3.0
 1.0  2.0  3.0
```

The effect of P on A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

Red arrows point from the handwritten matrix to the corresponding rows in the Julia output above.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = LU$$

$$A = P^{-1}LU = P^T LU$$

a different P than before ..

```
julia> P'*L*U
3x3 Matrix{Float64}:
 1.0  2.0  3.0
 4.0  5.0  6.0
 2.0  0.0 -3.0
```

```
julia> P=[0 1 0; 0 0 1; 1 0 0]
```

```
3×3 Matrix{Int64}:
```

```
0 1 0
```

```
0 0 1
```

```
1 0 0
```

```
julia> A=[ 1 2 3; 4 5 6; 2 0 -3]
```

```
3×3 Matrix{Int64}:
```

```
1 2 3
```

```
4 5 6
```

```
2 0 -3
```

```
julia> P*A
```

```
3×3 Matrix{Int64}:
```

```
4 5 6
```

```
2 0 -3
```

```
1 2 3
```