Idea solve systems of linear equations.

Solve $Ax = b$ for x

Here $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

Factorizations ... factor A into a product of easier to understand matrices

Algorithm

$A = LU$ — upper triangular; lower triangular — Gaussian elimination

$A = PLU$ — upper triangular; lower triangular; permutation matrix $P^{-1} = P^T$ — Gaussian elimination with row swaps

practical

$A = QR$ — upper triangular; orthogonal $Q^{-1} = Q^T$ — Gram-Schmidt orthogonalization

$A = S^{-1} D S$ — matrix of eigenvectors; diagonal matrix — not practical — eigenvalue eigenvector problem $\chi_A(\lambda) = \det(A - \lambda I) = 0$ to find the eigenvalues...

Iterative method of Francis.

$A = U \Sigma V^T$ — orthogonal; diagonal matrix; orthogonal matrix. — ?, Singular value decomposition Solve eigenvalue-eigenvector problem for $B = A^T A$.

eigenvectors of B normalized to unit length

Gaussian elimination

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$

do elimination steps to make a triangular matrix U.

$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$ $\quad r_2 \leftarrow r_2 - \frac{4}{1} r_1$ elimination.

$\quad r_3 \leftarrow r_3 - \frac{2}{1} r_1$

magnifying the error if multipliers in the elimination are greater than one

Pivots

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{-3} & -6 \\ 0 & -4 & -9 \end{bmatrix}$ $\quad r_3 \leftarrow r_3 - \frac{4}{3} r_2$

$-9 - \frac{4}{3}(-6) = -1$

$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & \frac{4}{3} & 1 \end{bmatrix}$

all the entries in L were greater than one...

Claim $A = LU$.

```
julia> U=[1 2 3; 0 -3 -6; 0 0 -1]
3×3 Matrix{Int64}:
 1   2   3
 0  -3  -6
 0   0  -1

julia> L=[1 0 0; 4 1 0; 2 4/3 1]
3×3 Matrix{Float64}:
 1.0  0.0      0.0
 4.0  1.0      0.0
 2.0  1.33333  1.0

julia> L*U
3×3 Matrix{Float64}:
 1.0  2.0   3.0
 4.0  5.0   6.0
 2.0  0.0  -3.0
```

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$

Perform row swaps so the multipliers in the elimination steps are less than 1.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 2 & 0 & -3 \end{bmatrix}$$

$r_2 \leftarrow r_2 - \frac{1}{4} r_1$

$r_3 \leftarrow r_3 - \frac{2}{4} r_1 = r_3 - \frac{1}{2} r_2$

$2 - \frac{1}{4} 5 = \frac{8-5}{4} = \frac{3}{4}$

$3 - \frac{1}{4} 6 = 3 - \frac{3}{2} = \frac{6-3}{2} = 3/2$

$-3 - \frac{1}{2} 6 = -3 - 3 = -6$

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & 3/4 & 3/2 \\ 0 & -5/2 & -6 \end{bmatrix}$$

so the next elimination step also has a small mult.

$r_2 \leftrightarrow r_3$

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & -5/2 & -6 \\ 0 & 3/4 & 3/2 \end{bmatrix}$$

$r_3 \leftarrow r_3 - \frac{(3/4)}{(-5/2)} r_2 = r_3 + \frac{3}{10} r_2$

$\frac{3}{2} + \frac{3}{10}(-6) = \frac{15-18}{10} = \frac{-3}{10}$

$$U = \begin{bmatrix} 4 & 5 & 6 \\ 0 & -5/2 & -6 \\ 0 & 0 & -3/10 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{3}{10} & 1 \end{bmatrix}$$

careful here because 1/2 and 1/4 go in different places because of the row swap $r_2 \leftrightarrow r_3$ done later

```
julia> L=[1 0 0; 1/2 1 0; 1/4 -3/10 1]
3×3 Matrix{Float64}:
 1.0    0.0  0.0
 0.5    1.0  0.0
 0.25  -0.3  1.0

julia> U=[4 5 6; 0 -5/2 -6; 0 0 -3/10]
3×3 Matrix{Float64}:
 4.0   5.0   6.0
 0.0  -2.5  -6.0
 0.0   0.0  -0.3

julia> L*U
3×3 Matrix{Float64}:
 4.0  5.0   6.0
 2.0  0.0  -3.0
 1.0  2.0   3.0
```

The effect of P on A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} \quad \xrightarrow{r_1 \leftrightarrow r_2} \quad \begin{matrix} a_2 \\ a_1 \\ a_3 \end{matrix} \quad \xrightarrow{r_2 \leftrightarrow r_3} \quad \begin{matrix} a_2 \\ a_3 \\ a_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = LU \qquad A = P^{-1}LU = P^{T}LU$$

a different P than before ..

```
julia> P'*L*U
3×3 Matrix{Float64}:
 1.0  2.0   3.0
 4.0  5.0   6.0
 2.0  0.0  -3.0
```

```
julia> P=[0 1 0; 0 0 1; 1 0 0]
3×3 Matrix{Int64}:
 0  1  0
 0  0  1
 1  0  0

julia> A=[ 1 2 3; 4 5 6; 2 0 -3]
3×3 Matrix{Int64}:
 1  2   3
 4  5   6
 2  0  -3

julia> P*A
3×3 Matrix{Int64}:
 4  5   6
 2  0  -3
 1  2   3
```