By induction means following the pattern.

Suppose $x_0$ satisfies $0 \lesssim x_0 < \xi_2$

Then

$$\alpha = \frac{1}{2}(x_0 + \xi_1) < \frac{1}{2}(\xi_2 + \xi_1) = \frac{1}{2}(1 + \cancel{\sqrt{c-1}} + 1 - \cancel{\sqrt{c-1}}) = 1$$

So

$$x_1 - \xi_1 = \frac{1}{2}(x_0 + \xi_1)(x_0 - \xi_1) = \alpha(x_0 - \xi_1)$$

$$|x_1 - \xi_1| \leq \alpha |x_0 - \xi_1|.$$

Try to continue. Need to show $0 \leq x_1 < \xi_2$.

Case $e_0 > 0$ then $e_{0+1} < |e_0|$ implies

$\boxed{x_{0+1} - \xi_1 < x_0 - \xi_1}$ and so $x_{0+1} < x_0 < \xi_2$

Case $e_0 < 0$ then $x_0 < \xi_1$

$$x_{0+1} = \frac{1}{2}(x_0^2 + c) < \frac{1}{2}(\xi_1^2 + c) = \frac{1}{2}(1 - 2\sqrt{1-c} + 1 - c + c) -$$

$$= 1 - \sqrt{1-c} < \xi_1 < \xi_2$$

In either case $0 \lesssim x_1 < \xi_2$. Also $x_1 < x_0$ in the first case so $\frac{1}{2}(x_1 + \xi_1) \leq \frac{1}{2}(x_0 + \xi_1)$

In the second case $\frac{1}{2}(x_1 + \xi_1) \leq \frac{1}{2}(\xi_1 + \xi_1)$.

$$\left.\begin{array}{l} \frac{1}{2}(x_1 + \xi_1) \leq \frac{1}{2}(x_0 + \xi_1) \\ \frac{1}{2}(x_1 + \xi_1) \leq \frac{1}{2}(\xi_1 + \xi_1) \end{array}\right\} \leq \gamma$$

where $\boxed{\gamma = \max\Big(\underbrace{\tfrac{1}{2}(x_0 + \xi_1)}_{\alpha}, \tfrac{1}{2}(\xi_1+\xi_1)\Big)}$

Thus $\alpha \leq \gamma$ and

$$|x_1 - \xi_1| \leq \alpha|x_0 - \xi_1| \leq \gamma|x_0 - \xi_1|$$

$x_1 + \xi_1 \leq \gamma$ (from the two cases)

Therefore

$$|x_2 - \xi_1| \leq \boxed{\tfrac{1}{2}|x_1 + \xi_1|}\,|x_1 - \xi_1| = \gamma^2|x_0 - \xi_1|$$

Following this pattern

$$|x_k - \xi_1| \leq \gamma^k |x_0 - \xi_1| \to 0 \quad \text{as } k \to \infty.$$

---

Newton's method: given $x_0$ define

$$x_{k+1} = g(x_k) \quad \text{where } g(x) = x - \frac{f(x)}{f'(x)}$$

Then $x_k$ is a sequence of approximations for the root $\xi$ where $f(\xi) = 0$ provided $x_0$ is close to $\xi$.

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Secant Method: Given $x_0, x_1$ define

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Idea is after solving $Ax=b$ for $x$ then plug the answer back in to get a residual error

$$r = A\tilde{x} - b$$

$\tilde{x}$ is approximation for exact answer..,

How well can the residual error be used to check whether $\tilde{x}$ is close to $x$?

The rounding errors when computing $r$ determine whether the computer can even tell the difference between $\tilde{x}$ and the true solution $x$.

Given a matrix $A$ how well does computing $r$ allow us to check whether our approximation is correct.

How big is $r$? how much does $x$ and $\tilde{x}$ differ?

Use norms: Suppose $r \in \mathbb{R}^n$

Euclidean norm

$$\|r\|_2 = \sqrt{r \cdot r} = \sqrt{\sum_{i=1}^{n} r_i^2}$$

Maximum norm

$$\|r\|_\infty = \max\{|r_1|, |r_2|, \ldots, |r_n|\}$$

1-norm

$$\|r\|_1 = |r_1| + |r_2| + \cdots + |r_n|$$

What's a norm?

**Definition 2.6** *Suppose that $\mathcal{V}$ is a linear space over the field $\mathbb{R}$ of real numbers. The nonnegative real-valued function $\|\cdot\|$ is said to be a* **norm** *on the space $\mathcal{V}$ provided that it satisfies the following axioms:*

❶ $\|v\| = 0$ *if, and only if,* $v = 0$ *in* $\mathcal{V}$*;*
❷ $\|\lambda v\| = |\lambda|\,\|v\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $v$ *in* $\mathcal{V}$*;*
❸ $\|u + v\| \leq \|u\| + \|v\|$ *for all* $u$ *and* $v$ *in* $\mathcal{V}$ *(the triangle inequality).*

Question: do the Euclidean, 1-norm and maximum norms satisfy these three properties? Yes. How? Why?

Many more norms for $p \geq 1$

$$\|r\|_p = \sqrt[p]{\sum |r_i|^p} .$$