For $p \in [1, \infty)$ define.

$$\|r\|_p = \sqrt[p]{\sum_{i=1}^{n} |r_i|^p}$$

$$\|r\|_\infty = \max\{|r_1|, |r_2|, \dots, |r_n|\}$$

<u>Claim</u> $\lim_{p \to \infty} \|r\|_p = \|r\|_\infty$ (see this later).

$$\|r\|_\infty = \max\{|r_1|, \dots, |r_n|\} = |r_k| \quad \text{for some } k.$$

$$= \sqrt[p]{|r_k|^p} \leq \sqrt[p]{|r_1|^p + \cdots + |r_n|^p} = \|r\|_p$$

Therefore

$$\|r\|_\infty \leq \|r\|_p.$$

$$\|r\|_p = \sqrt[p]{\underbrace{|r_1|^p + \cdots + |r_n|^p}_{\substack{\text{n terms each} \\ \text{is less than} \\ \max |r_i|^p}}} \leq \sqrt[p]{n \max |r_i|^p}$$

$$= \sqrt[p]{n \|r\|_\infty^p} = \sqrt[p]{n}\, \|r\|_\infty$$

Therefore

$$\|r\|_p \leq \sqrt[p]{r}\, \|r\|_\infty$$

In summary

$$\|r\|_\infty \le \|r\|_p \le \sqrt[p]{n}\,\|r\|_\infty$$

assuming $r \neq 0$ then take $p \to \infty$

$$1 \le \frac{\|r\|_p}{\|r\|_\infty} \le \sqrt[p]{n} = n^{1/p} = e^{\frac{1}{p}\ln n} \to e^0 = 1$$

Therefore the ratio in the middle gets squeezed

$$\lim_{p\to\infty} \frac{\|r\|_p}{\|r\|_\infty} = 1$$

Or

$$\lim_{p\to\infty} \|r\|_p = \|r\|_\infty$$

This explains why $\|r\|_\infty$ is used for the max. norm.

Last time we showed

$$\left|\sum u_i v_i\right| \le \|u\|_p \|v\|_q \quad \text{when } \frac{1}{p}+\frac{1}{q}=1$$

what's next is to show the p-norm satisfies the triangle inequality.

Theorem 2.6 **(Minkowski's inequality)** *Let* $1 \le p \le \infty$ *and* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$*. Then,*

$$\|\boldsymbol{u}+\boldsymbol{v}\|_p \le \|\boldsymbol{u}\|_p + \|\boldsymbol{v}\|_p .$$

<u>Proof:</u> Let $p>1$ since $p=1$ is already done...

main idea

$$\|u+v\|_p^p = \sum_{i=1}^n |u_i+v_i|^p = \boxed{\sum_{i=1}^n |u_i+v_i|^{p-1} \underbrace{|u_i+v_i|}}$$

apply triangle inequality to abs. value.

$$\le \sum_{i=1}^n |u_i+v_i|^{p-1}\left(|u_i|+|v_i|\right)$$

$$= \underbrace{\sum_{i=1}^n |u_i+v_i|^{p-1}|u_i|}_{\text{apply hölder ineq.}} + \underbrace{\sum_{i=1}^n |u_i+v_i|^{p-1}|v_i|}_{\text{apply hölder ineq.}}$$

now just simplify.

$$\sum_{i=1}^n |u_i+v_i|^{p-1}|u_i| = \sum_{i=1}^n a_i b_i \le \|a\|_q \|b\|_p \quad \text{where } \frac{1}{q}+\frac{1}{p}=1$$

$$= \sqrt[q]{\sum_{i=1}^n |a_i|^q}\ \sqrt[p]{\sum_{i=1}^n |b_i|^p} = \sqrt[q]{\sum_{i=1}^n |u_i+v_i|^{q(p-1)}}\ \underbrace{\sqrt[p]{\sum_{i=1}^n |u_i|^p}}_{\|u\|_p}$$

$$\frac{1}{q}+\frac{1}{p}=1, \quad \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}, \quad q=\frac{p}{p-1}, \quad q(p-1)=p$$

Thus,

$$\sum_{i=1}^n |u_i+v_i|^{p-1}|u_i| \le \sqrt[q]{\sum_{i=1}^n |u_i+v_i|^p}\ \sqrt[p]{\sum_{i=1}^n |u_i|^p}$$

also

$$\sum_{i=1}^{n} |u_i+v_i|^{p-1} |v_i| \leq \sqrt[q]{\sum_{i=1}^{n} |u_i+v_i|^p} \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}$$

Thus

$$\|u+v\|_p^p \leq \sum_{i=1}^{n} |u_i+v_i|^{p-1} |u_i| + \sum_{i=1}^{n} |u_i+v_i|^{p-1} |v_i|$$

$$\leq \sqrt[q]{\sum_{i=1}^{n} |u_i+v_i|^p} \left(\underbrace{\sqrt[p]{\sum_{i=1}^{n} |u_i|^p}}_{\|u\|_p} + \underbrace{\sqrt[p]{\sum_{i=1}^{n} |v_i|^p}}_{\|v\|_p} \right)$$

divide to other side.

$$\sqrt[q]{\sum_{i=1}^{n} |u_i+v_i|^p} = \sqrt[q]{\left(\sqrt[p]{\sum_{i=1}^{n} |u_i+v_i|^p}\right)^p} = \left(\sqrt[p]{\sum_{i=1}^{n} |u_i+v_i|^p}\right)^{p/q}$$

recall

$$\frac{1}{q} + \frac{1}{p} = 1, \quad \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}, \quad \frac{p}{q} = p-1$$

$$\sqrt[q]{\sum_{i=1}^{n} |u_i+v_i|^p} = \Bigg(\underbrace{\sqrt[p]{\sum_{i=1}^{n} |u_i+v_i|^p}}_{\|u+v\|_p}\Bigg)^{p-1} = \|u+v\|_p^{p-1}$$

$$\|u+v\|_p^p \leq \|u+v\|_p^{p-1} \left(\|u\|_p + \|v\|_p \right)$$

Dividing yields

$$\frac{\|u+v\|_p^p}{\|u+v\|_p^{p-1}} \leq \|u\|_p + \|v\|_p$$

or $\|u+v\|_p \leq \|u\|_p + \|v\|_p$, The triangle inequality...

Now what about the norm of a matrix?

Properties of a matrix norm: Let $A, B \in \mathbb{R}^{n\times n}$.

1. $\|A\| = 0$ *if, and only if,* $A = 0$ *in* $\mathcal{V}$;
2. $\|\lambda A\| = |\lambda|\,\|A\|$ *for all* $\lambda \in \mathbb{R}$ *and all* v *in* $\mathcal{V}$;
3. $\|A+B\| \leq \|A\| + \|B\|$ *for all* u *and* v *in* $\mathcal{V}$ *(the triangle inequality).*
4. $\|Av\| \leq \|A\|\,\|v\|$, for all $v \in \mathbb{R}^n$.

Natural matrix norm induced by a vector norm.

Definition

$$\underbrace{\|A\|_p}_{\text{matrix p-norm}} = \max\left\{ \underbrace{\|Av\|_p}_{\text{vector p-norm}} : \underbrace{\|v\|_p}_{\text{vector p-norm}} = 1 \right\}.$$

does this maximum exist? Need the set $\{\|Av\|_p : \|v\|_p = 1\}$ to be closed and bounded. YES. it exists...

How to find the Matrix norm?

We will work 3 special cases: $p=1$, $p=\infty$ and $p=2$.
That leaves the other values of p.

EXTRA CREDIT.

```
julia> using LinearAlgebra

julia> A=rand(3,3)
3×3 Matrix{Float64}:
 0.727132  0.294341  0.165471
 0.262967  0.606946  0.745738
 0.744013  0.961251  0.116066

julia> opnorm(A,2)
1.6187518637590053
```

$= \|A\|_2$

```
julia> opnorm(A,1)
1.8625378061167959
```

$= \|A\|_1$

```
julia> opnorm(A,Inf)
1.8213304699268678
```

$= \|A\|_\infty$

```
julia> opnorm(A,3)
ERROR: ArgumentError: invalid p-norm p=3. Valid: 1, 2, Inf
Stacktrace:
 [1] opnorm(A::Matrix{Float64}, p::Int64)
   @ LinearAlgebra /buildworker/worker/package_linux64/build/usr/share/julia/stdlib/v1.6/LinearAlgebra/src/generic.jl:773
 [2] top-level scope
   @ REPL[6]:1
```

Note Julia only treats these 3 special cases as well...

EXTRA CREDIT

For extra credit please read

- Nicholas Higham, Estimating the matrix p-norm, *Numer. Math.*, vol 62, 1992, pp. 539-555.

and implement the algorithm to find the Matrix p-norm in Julia. Though it may not help much, there is a C++ implementation of this algorithm for GNU Octave. I do not know of any native Julia code which does the same.

also finding someone else's Julia code and testing it to see it work is also worth points...

Let consider $\|A\|_1 = \max\{\underbrace{\|Ax\|_1}_{\text{what is this?}} : \|x\|_1 = 1\}$

$$[Ax]_i = \sum_{j=1}^{m} A_{ij} x_j$$

$$\|Ax\|_1 = \sum_{i=1}^{n} |[Ax]_i| = \sum_{i=1}^{n} \left| \sum_{j=1}^{m} A_{ij} x_j \right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} |A_{ij}||x_j| = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} |A_{ij}| \right)}_{\text{for each } j \text{ this is a different value}} |x_j|$$

Let $M = \max_{j=1}^{n} \sum_{i=1}^{n} |A_{ij}|$

$$\|Ax\|_1 \leq \sum_{j=1}^{n} M |x_j| = M \sum_{j=1}^{n} |x_j| = M \|x\|_1$$

Claim $\|A\|_1 = M$

finish this next time.

right now we have the bound $\|A\|_1 \leq M$
what's left is to show $\|A_1\| \geq M$.