Summary to approximate $\|A\|_2$

① Choose a random unit vector x_0

② $y_k = B x_k$

③ $\lambda \approx y_k \cdot x_k$

④ $x_{k+1} = y_k / \|y_k\|$

repeat...

Can this algorithm be used to find $\|A^{-1}\|_2$.

Could set $B = (A^{-1})^T (A^{-1})$ and apply the above algorithm, but that involves knowing A^{-1}.

Alternatives...

First suppose that A^{-1} exists, otherwise $\|A^{-1}\|_2$

doesn't make sense otherwise...

Assume $\det(A) \neq 0$

Since $C = AA^T$ then note $C^T = C$ still holds.

$\det(C) = \det A \det A^T = (\det A)^2 \neq 0$ ✓

Let λ_i be the eigenvalues of C and ξ_i the orthonormal basis of eigenvectors

$$\det C = \lambda_1 \lambda_2 \cdots \lambda_n \neq 0$$

☑ So all eigenvalues $\lambda_i > 0$

Remember $C^T = C$ implies $\lambda_i \in \mathbb{R}$ and

$$C\xi_i \cdot \xi_i = A A^T \xi_i \cdot \xi_i = A^T \xi_i \cdot A^T \xi_i = \|A^T \xi_i\|^2 \geq 0$$

$$\|$$

$$\lambda_i \xi_i \cdot \xi_i = \lambda_i \, \xi_i \cdot \xi_i = \lambda_i$$

Thus $\lambda_i = \|A^T \xi_i\|^2 \geq 0$ Combined

Thus $\lambda_i > 0$

$$Q = \left[\xi_1 \middle| \xi_2 \middle| \ldots \middle| \xi_2 \right]$$ then $Q^T Q = I$ and since Q is square then $Q^{-1} = Q^T$.

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$$

$CQ = QD$ or $C = QDQ^{-1}$.

$B = A^T A$ so $B^{-1} = (A^T A)^{-1} = A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T$

$C = A A^T$ so $C^{-1} = (A A^T)^{-1} = (A^T)^{-1} A^{-1} = (A^{-1})^T A^{-1}$

$C^{-1} x \cdot x = (A^{-1})^T A^{-1} x \cdot x = A^{-1} x \cdot A^{-1} x = \| A^{-1} x \|^2$

① Choose a random unit vector x_0

② $y_k = C^{-1} x_k$ (solve for y_k so that $C y_k = x_k$)

in Julia $y_k = C \backslash x_k$

③ $\lambda \approx y_k \cdot x_k$

④ $x_{k+1} = y_k / \| y_k \|$

The largest eigenvalue of C^{-1} so $\| A^{-1} \| = \sqrt{\lambda}$

```
julia> R10=rand(10,10);

julia> C=R10*R10';
       B = R10' * R10;
julia> using LinearAlgebra
```

```
julia> xk=rand(10); xk=xk/norm(xk)
       for k=1:10
           yk=C\xk
           println("lambda = ",yk'*xk)
           xk=yk/norm(yk)
       end
lambda = 22.322810516739516
lambda = 1900.45492940512
lambda = 1924.5802738767622
lambda = 1924.5985880987423
lambda = 1924.5986019111508
lambda = 1924.59860192157 2
lambda = 1924.5986019215798
lambda = 1924.5986019215795
lambda = 1924.5986019215777
lambda = 1924.5986019215793
```

approx of Largest eigenvalue of C^{-1}

```
julia> sqrt(1924.5986019215793)
43.87024734283566
```

$\| A^{-1} \| \approx 43.87 \ldots$

Check with the built in function...

```
julia> opnorm(inv(R10))
43.87024734284606
```

Remark: That solving an $n \times n$ linear system 10 times is much less work that inverting a $n \times n$ matrix when n is large...

Note: finding the inverse of an $n \times n$ matrix is equivalent to solving n linear systems...

The condition number:

$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

these through forward iterations of $B = A^T A$

these through backward iterations of $C = AA^T$.

What happens if use B in place of C?

1. Choose a random unit vector x_0

2. $y_k = B^{-1} x_k$ (solve for y_k so that $B y_k = x_k$)

   in Julia $y_k = B \backslash x_k$

3. $\lambda \approx y_k \cdot x_k$

4. $x_{k+1} = y_k / \|y_k\|$

```
julia> B=R10'*R10;

julia> xk=rand(10); xk=xk/norm(xk)
       for k=1:10
           yk=B\xk
           println("lambda = ",yk'*xk)
           xk=yk/norm(yk)
       end
lambda = 10.691182632185177
lambda = 1920.580590479082
lambda = 1924.5956339384934
lambda = 1924.5985996823217
lambda = 1924.5986019194495
lambda = 1924.598601921137
lambda = 1924.598601921138
lambda = 1924.5986019211377
lambda = 1924.5986019211384
lambda = 1924.5986019211389
```

Converged to the same λ and with C.

How are the eigenvalues of a matrix changed under different algebraic operations?

Suppose $B\xi = \lambda\xi$ and B is invertible.

$$B^{-1}B\xi = B^{-1}\lambda\xi$$

$$B^{-1}\lambda\xi = \xi$$

$$\lambda B^{-1}\xi = \xi.$$

$$B^{-1}\xi = \frac{1}{\lambda}\xi.$$

Conclusion: The eigenvalues of B^{-1} are $\frac{1}{\lambda}$

where λ are the eigenvalues of B.

Also the eigenvectors are the same ...

Suppose $B\xi = \lambda\xi$ and consider $B + \alpha I$

$$(B+\alpha I)\xi = B\xi + \alpha I \xi = \lambda\xi + \alpha\xi = (\lambda+\alpha)\xi.$$

Conclusion: The eigenvalues of $B+\alpha I$ are $\lambda+\alpha$
where λ are the eigenvalues of B.
Also the eigenvectors are the same.

Question: are the eigenvalues of A and A^T the same?
are the eigenvectors of A and A^T the same?

Recall A^T is exactly the matrix M so that.

(*) $\quad Ax \cdot y = x \cdot My \quad$ for all vectors x, y.

If you stop thinking about A^T as in terms of reflecting the entries in a table of numbers and instead as (*).
Then maybe answering the question is easier.