Definition 5.5 *Suppose that $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$. The* **Gerschgorin discs** D_i, $i = 1, 2, \ldots, n$, *of the matrix A are defined as the closed circular regions*

large terms on diagonal

$$D_i = \{z \in \mathbb{C} \colon |z - a_{ii}| \leq R_i\} \tag{5.17}$$

in the complex plane, where

$$R_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \tag{5.18}$$

size of off diagonal terms for row i

is the radius of D_i.

Theorem 5.4 (Gerschgorin's Theorem) *Let $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$. All eigenvalues of the matrix A lie in the region $D = \bigcup_{i=1}^{n} D_i$, where D_i, $i = 1, 2, \ldots, n$, are the Gerschgorin discs of A defined by (5.17), (5.18).*

Theorem 5.5 (Gerschgorin's Second Theorem) *Let $n \geq 2$. Suppose that $1 \leq p \leq n-1$ and that the Gerschgorin discs of the matrix $A \in \mathbb{C}^{n\times n}$ can be divided into two disjoint subsets $D^{(p)}$ and $D^{(q)}$, containing p and $q = n - p$ discs respectively. Then, the union of the discs in $D^{(p)}$ contains p of the eigenvalues, and the union of the discs in $D^{(q)}$ contains $n - p$ eigenvalues. In particular, if one disc is disjoint from all the others, it contains exactly one eigenvalue, and if all the discs are disjoint then each disc contains exactly one eigenvalue.*

$I \subseteq \{1, \ldots, n\}$ such that $\operatorname{card} I = p$

$D^{(p)} = \bigcup_{i\in I} D_i \qquad D^{(q)} = \bigcup_{i\notin I} D_i$

By hypothesis $D^{(p)} \cap D^{(q)} = \emptyset$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \qquad B_0 = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$

$$B(\epsilon) = \epsilon A + (1-\epsilon) B_0 = \begin{bmatrix} a_{11} & \epsilon a_{12} & \cdots & \epsilon a_{1n} \\ \epsilon a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ \epsilon a_{n1} & \epsilon a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

ϵ in front of off diagonal terms

the diagonal terms unchanges...

Thus $B(1) = A$ and $B(0) = B_0$

We know exactly what the eigenvalues of $B(0)$ are.

Definition 5.5 *Suppose that $n \geq 2$ and $B(\epsilon) \in \mathbb{C}^{n\times n}$. The* **Gerschgorin discs** D_i, $i = 1, 2, \ldots, n$, *of the matrix A are defined as the closed circular regions*

$$D_i^{(\epsilon)} = \{z \in \mathbb{C}\colon |z - a_{ii}| \leq R_i(\epsilon)\} \tag{5.17}$$

in the complex plane, where

$$R_i^{(\epsilon)} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \epsilon |a_{ij}| . \tag{5.18}$$

is the radius of D_i.

In the case of B_0 then $\epsilon = 0$ and the discs have radius 0.

Note $D^{(p)}(0) = \bigcup_{i \in I} D_i(0)$ contains exactly p eigenvalues

Want to increase the ε.

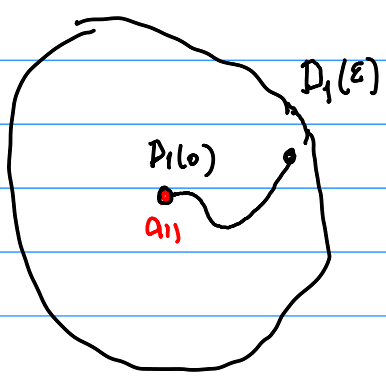
$$\chi_{B(\varepsilon)}(\lambda) = \det\left(B(\varepsilon) - \lambda I\right)$$

This is a polynomial of degree n with coefficients that depend continuously on ε.

$$\chi_{B(\varepsilon)}(\lambda) = \det \begin{bmatrix} a_{11}-\lambda & \varepsilon a_{12} & \cdots & \varepsilon a_{1n} \\ \varepsilon a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ \varepsilon a_{n1} & \varepsilon a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Also the solutions of $\chi_{B(\varepsilon)}(\lambda) = 0$ also depend continuously on ε.

Starting at $\varepsilon = 0$ and then increasing ε the eigenvalues of $B(\varepsilon)$ trace out continuous paths in the complex plane.



This happens continuously for every eigenvalue

Thus $D^{(p)}(\varepsilon) = \bigcup_{i \in I} D_i(\varepsilon)$ still contains at least p eigenvalues.

$$D^{(p)} = \bigcup_{i \in I} D_i$$ contains at least p eigenvalues

There aren't any more because

$$D^{(q)} = \bigcup_{i \notin I} D_i \quad \text{contains at least } q \text{ eigenvalues}$$

Since $D^{(p)} \cap D^{(q)} = \emptyset$ and there are only $p+q=n$ eigenvalues in total, then $D^{(p)}$ must have exactly p eigenvalues.

End of the discussion of eigenvalues...

Note another algorithm Eigenvalue algorithm of Francis is at the end of the chapter. Same Idea a Jacobi except the sequence of orthogonal transformations to make the $A^{(k)}$'s is different...

<u>Least Squares</u>: Idea: minimize $\|Ax-b\|$ to approximate the solution to $Ax=b$.

<u>Math 330</u> we factor $A = QR$ ← upper triangular.

orthogonal matrix

then solve $Rx = Q^T b$ for x

and this is the minimizer of $\|Ax-b\|$

How to factor A into QR?

In Math 330 learned Gram-Schmidt algorithm to find R and subsequently Q.

Use column operations

Recall Gaussian elimination uses row operations to find $A = PLU$

upper triangular.

permutation of the rows

Lower triangular

Numerically pivoting allowed to control rounding error... swapped rows to make the pivot have largest magnitude.

No similar flexibility in Gram-Schmidt... We can't swap columns..

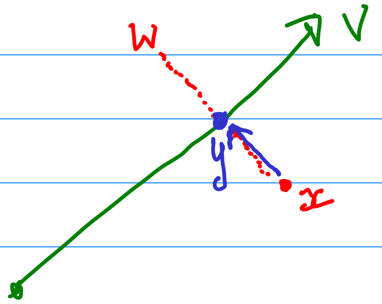
Gram-schmidt doesn't have additional flexibility in how you do it that can be used to reduce rounding error.

Idea consider a sequence of orthogonal matrices that multiply together to find Q then get R.

Used plane rotations for the orthogonal matrices in Jacobi algorithm... we'll use reflections to find QR.

Householder Reflectors

defined by a unit vector $v \in \mathbb{R}^n$



reflect x about the line given by v to find w.

Then

$$y - x = w - y$$

Using dot product, y is the projection of x onto v.

$$y = (v \cdot x)v$$

$$w = (w - y) + y = y - x + y = 2y - x$$

$$= 2(v \cdot x)v - x = 2\underbrace{(v^T x)}_{\text{scalar}}v - x$$

write in terms of a matrix

$$= 2v(v^T x) - x = 2(vv^T)x - x = \underbrace{(2vv^T - I)}_{\text{matrix}}x$$

associative property

Define

$$H = I - 2vv^T$$

Claim H is an orthogonal matrix.

$$H^TH = (I - 2vv^T)^T(I - 2vv^T)$$

$$= (I^T - (2vv^T)^T)(I - 2vv^T)$$

$$= (I - 2v^{TT}v^T)(I - 2vv^T)$$

$$= (I - 2vv^T)(I - 2vv^T)$$

Note $H^T = H$
symmetric

$$= I - 2vv^T - 2vv^T + 4vv^Tvv^T$$

$$= I - 2vv^T - 2vv^T + 4v(v \cdot v)v^T$$

since v is unit vector
$v \cdot v = 1$

$$= I - 2\cancel{vv^T} - 2\cancel{vv^T} + 4\cancel{vv^T} = I.$$

Make Q out of a product of these H's to find the QR factorization of A.