Exercise 1.4.1

The formulas

$$f(x) = \frac{1-\cos(x)}{\sin(x)}, \quad g(x) = \frac{2\sin^2(x/2)}{\sin(x)},$$

are mathematically equivalent, but they suggest evaluation algorithms that can behave quite differently in floating point.

$$\boxed{1-\cos x = 2\sin^2(x/2)} \quad ?$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a \cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$a = \frac{x}{2} \quad b = \frac{x}{2}$$

$$\boxed{\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

Pythagorean theorem:

$$\cos^2\frac{x}{2} + \sin^2\frac{x}{2} = 1$$

$$-\quad \cos^2\frac{x}{2} + \sin^2\frac{x}{2} = -\cos x$$

$$2\sin^2\frac{x}{2} = 1 - \cos x$$

Thus

$$\underbrace{\frac{1-\cos x}{\sin x}}_{f(x)} = \underbrace{\frac{2\sin^2 x/2}{\sin x}}_{g(x)}$$

due to rounding an approximation error these are not equal on a computer.

$$K_f(x) = |f'(x)|\left|\frac{x}{f(x)}\right| \qquad K_g(x) = |g'(x)|\left|\frac{x}{g(x)}\right|$$

$$g'(x) = \frac{2(\sin^{x}/_2)(\cos^{x}/_2)(1/_2)\sin x - 2\sin^2 x/_2 \cos x}{(\sin x)^2} = ?$$

$$f'(x) = \frac{\sin^2 x - (1-\cos x)\cos x}{(\sin x)^2} = \frac{\sin^2 x - \cos x + \cos^2 x}{(\sin x)^2} = \frac{1-\cos x}{(\sin x)^2}$$

$$K_f(x) = \left|\frac{1-\cos x}{(\sin x)^2}\ \frac{x}{\frac{1-\cos x}{\sin x}}\right| = \left|\frac{x}{\sin x}\right|$$

$$\lim_{x\to 0} K_f(x) = \lim_{x\to 0}\left|\frac{x}{\sin x}\right| = \left|\lim_{x\to 0}\frac{x}{\sin x}\right| = 1$$

by L'hospital's rule

$$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{1} = 1$$

Small condition number

$$\left|\frac{f(\tilde{x}) - f(x)}{f(x)}\right| \approx \varepsilon K_f(x)$$

where ε is the relative error in $\tilde{x}$

relative error in output

relative error in the input

```
julia> f(x)=(1-cos(x))/sin(x)
f (generic function with 1 method)

julia> g(x)=2*sin(x/2)^2/sin(x)
g (generic function with 1 method)

julia> f(1)
0.5463024898437905

julia> g(1)
0.5463024898437905

julia> f(1)==g(1)
true
```

exactly the same answer... miracle...

```
julia> f(0.1)
0.050041708375538244

julia> g(0.1)
0.05004170837553879
```

~~14 digits agree~~ (really only 13)

differences because of rounding...

```
julia> f(0.01)
0.005000041666708478

julia> g(0.01)
0.0050000416670833376
```

12 digits agree

```
julia> f(0.001)
0.0005000000416588394

julia> g(0.001)
0.0005000000416666708
```

11 digits agree

```
julia> f(0.0001)
4.99999977945978e-5

julia> g(0.0001)
5.000000004166667e-5
```

8 digits agree...

```
julia> f(0.00001)
5.000000413785189e-6

julia> g(0.00001)
5.000000000041667e-6
```

7 digits agree

```
julia> f(0.000001)
5.000444502912538e-7

julia> g(0.000001)
5.000000000000416e-7
```

4 digits

```
julia> f(0.0000001)
4.996003610813213e-8

julia> g(0.0000001)
5.0000000000000044e-8
```

3 digits agree...

```
julia> f(0.00000001)
0.0

julia> g(0.00000001)
5.0e-9
```

limiting value

not at the limit... don't want to round off to zero before it should... want to know how the function approaches zero...

all digits different

subtraction of two nearly equal numbers.

(loss of precision)

$$\underbrace{\frac{1-\cos x}{\sin x}}_{f(x)} = \underbrace{\frac{2\sin^2 x/2}{\sin x}}_{g(x)}$$

Step at a time to see what happens

x 10^{-6}

$\cos x$

1 2 3 4 5 6 7 8 9 0 1 2 3

```
0.9999999999995
```

$1-\cos x$

```
5.000444502911705e-13
```

precision left ? wrong

99

1.~~00000000~~0 0

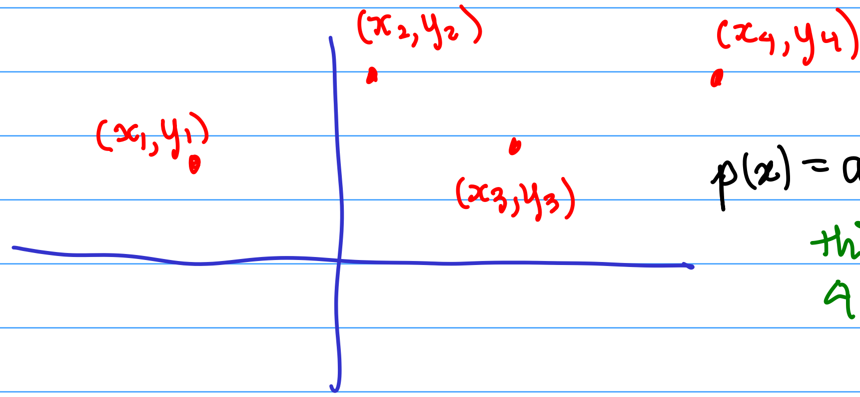
```
0.9999999999995
```

.00 - - 05

L'hospital's rule

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = 0$$

Polynomial interpolation



$p(x) = ax^3 + bx^2 + cx + d$

third degree has 4 paraters.

Idea is to solve for a, b, c and d such that

$$p(x_1) = y_1 \quad p(x_2) = y_2 \quad p(x_3) = y_3 \quad p(x_4) = y_4$$

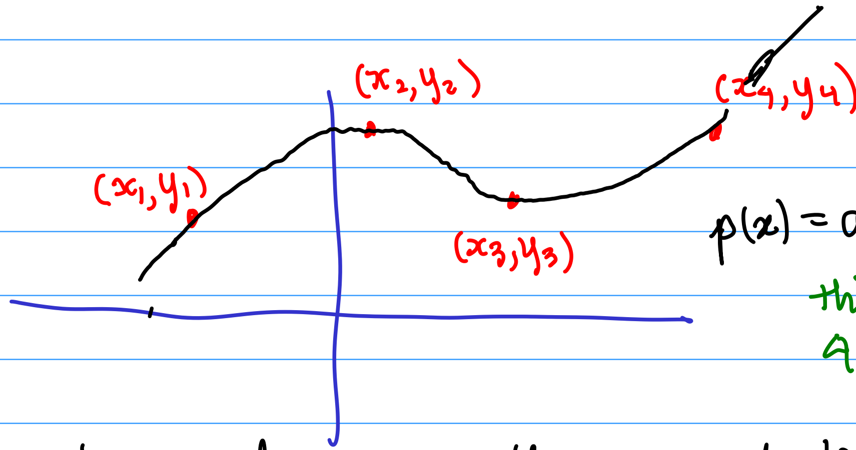
Note 4 equations with 4 unknowns... In fact they are linear equations...

$$a x_1^3 + b x_1^2 + c x_1 + d = y_1$$
$$a x_2^3 + b x_2^2 + c x_2 + d = y_2$$
$$a x_3^3 + b x_3^2 + c x_3 + d = y_3$$
$$a x_4^3 + b x_4^2 + c x_4 + d = y_4$$

(Vandermonde matrix)

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Solve system of linear equation to get polynomial



$p(x) = ax^3 + bx^2 + cx + d$

third degree has 4 paraters.

polynomial passes through all the points ...

use this to represent functions by polynomials...

Polynomials can approximate arbitrary functions.