```
julia> A=randn(4,4)
4×4 Matrix{Float64}:
 -0.316503  -0.0893162  -0.205544    0.927764
  0.496956  -0.51921     1.00072    -0.31696
 -0.92511    1.39363    -0.722981   -1.01798
 -0.72828   -0.438684   -0.0222849   2.26917

julia> A=round.(A*20)
4×4 Matrix{Float64}:
  -6.0   -2.0   -4.0   19.0
  10.0  -10.0   20.0   -6.0
 -19.0   28.0  -14.0  -20.0
 -15.0   -9.0   -0.0   45.0
```

$19+6+20+45=90$

$4+20+14=38$

$2+10+28+9=49$

$6+10+19+15=50$

```
julia> 6+10+19+15
50

julia> 2+10+28+9
49

julia> 4+20+14
38

julia> 19+6+20+45
90
```

$$\boxed{\|A\|_1 = \max\left\{\sum_{i=1}^{m} |A_{ij}| : j=1,\dots,n\right\}} = \max\{50, 49, 38, 90\} = 90$$

```
julia> using LinearAlgebra

julia> opnorm(A,1)
90.0
```

With respect to the Euclidean vector norm...

$\|A\|_2 =$

```
julia> opnorm(A,2)
57.15499779151479

julia> opnorm(A)
57.15499779151479
```

$= \max\{\|Ax\|_2 : \|x\|_2 = 1\}$.

We'll work on $\|A\|_2$ soon... not today...

Condition number:

$$\lim_{\tilde{x}\to x} \left|\frac{f(\tilde{x})-f(x)}{f(x)}\right| \Big/ \left|\frac{\tilde{x}-x}{x}\right| = K_f(x)$$

If $x \in \mathbb{R}$ then $\left|\frac{\tilde{x}-x}{x}\right| = \frac{|\tilde{x}-x|}{|x|}$

Then approximately

$$\frac{|f(\tilde{x})-f(x)|}{|f(x)|} \approx K_f(x) \underbrace{\left|\frac{\tilde{x}-x}{x}\right|}_{\text{relative error in } x. \text{ bounded by } \frac{1}{2}\varepsilon_{mach}.} \leq K_f(x)\, \frac{1}{2}\varepsilon_{mach}.$$

Same thing for solving $\boxed{Ax=b}$:

b input and $\boxed{f(b) = A^{-1}b = x}$ as output

$$\lim_{\tilde{b}\to b} \underbrace{\frac{\|f(\tilde{b})-f(b)\|}{\|f(b)\|}}_{\text{relative error in the answer.}} \Big/ \underbrace{\frac{\|\tilde{b}-b\|}{\|b\|}}_{\text{relative error in the vector } b.}$$

why we write as quotient of norms...

If $b \in \mathbb{R}^n$ then $\frac{\tilde{b}-b}{b}$ doesn't make sense ← can't divide by a vector

$$\frac{\|f(\tilde{b})-f(b)\|}{\|f(b)\|} \Big/ \frac{\|\tilde{b}-b\|}{\|b\|} = \frac{\overbrace{\|A^{-1}\tilde{b}-A^{-1}b\|}^{\text{factor}}}{\|A^{-1}b\|} \cdot \frac{\|b\|}{\|\tilde{b}-b\|} = \frac{\|A^{-1}(\tilde{b}-b)\|}{\|x\|} \cdot \frac{\|Ax\|}{\|\tilde{b}-b\|}$$

Properties of Matrix norm (not matter what vector norm)

$$\|A\|_p = \max\{\|Ax\|_p : \|x\|_p = 1\}$$

If $\|x\|_p = 1$ then $\|Ax\|_p \leq \|A\|_p$

Consider any vector $x \in \mathbb{R}^n$. Then $\frac{x}{\|x\|_p}$ is a unit vector with respect to the p-vector norm.

Since $\left\| \frac{x}{\|x\|_p} \right\|_p = \frac{\|x\|_p}{\|x\|_p} = 1$ then $\left\| A \frac{x}{\|x\|_p} \right\|_p \leq \|A\|_p$

mult this through

Therefore $\|Ax\|_p \leq \|A\|_p \|x\|_p$

for any $x \in \mathbb{R}^n$.

Continuing with the algebra from before

$$\ldots = \frac{\|A^{-1}(\tilde{b}-b)\|}{\|x\|} \cdot \frac{\|Ax\|}{\|\tilde{b}-b\|} \leq \frac{\|A^{-1}\| \, \cancel{\|\tilde{b}-b\|}}{\cancel{\|x\|}} \cdot \frac{\|A\| \, \cancel{\|x\|}}{\cancel{\|\tilde{b}-b\|}} = \underline{\|A^{-1}\| \, \|A\|}$$

for any matrix norm and compatible vector norm...

Therefore

$$\frac{\|f(\tilde{b})-f(b)\|}{\|f(b)\|} \Big/ \frac{\|\tilde{b}-b\|}{\|b\|} \leq \underbrace{\|A^{-1}\| \, \|A\|}_{\text{condition number of the matrix}} = K$$

or

$$\frac{\|f(\tilde{b})-f(b)\|}{\|f(b)\|} \leq \|A^{-1}\| \, \|A\| \frac{\|\tilde{b}-b\|}{\|b\|} \leq K \frac{\|\tilde{b}-b\|}{\|b\|}$$

Since $\frac{\|f(\tilde{b})-f(b)\|}{\|f(b)\|} = \frac{\|A^{-1}\tilde{b} - A^{-1}b\|}{\|A^{-1}b\|} = \frac{\|\tilde{x}-x\|}{\|x\|}$

Then

$$\frac{\|\tilde{x}-x\|}{\|x\|} \leq K \frac{\|\tilde{b}-b\|}{\|b\|}$$

Recall $Ax=b$ and $\tilde{x}$ is an approximation of x (found on a computer) and $\tilde{b} = A\tilde{x}$

what you get by plugging the approx. back into the problem

Interpretation

$\tilde{b} - b = A\tilde{x} - b =$ residual error when plugging the approximation into the equation it was supposed to satisfy...

If K is large it is difficult to tell how much error there was in $\tilde{x}$ from the residual error.

Case study: Finding polynomials that approximate other functions.

Given $f: [0,1] \to \mathbb{R}$ want to find $p(x) = \underbrace{\sum_{j=1}^{n} a_j x^{j-1}}_{\text{polynomial of degree } n-1}$

such that $\int_0^1 |f(x) - p(x)|^2 \, dx$ is minimized.

solve for the a_j's.

$$\frac{d}{da_i} \int_0^1 \left(f(x) - \sum_{j=1}^{n} a_j x^{j-1}\right)^2 dx = 0 \quad \text{solve for } a_i$$

$$\frac{d}{da_i} \int_0^1 \left(f(x) - \sum_{j=1}^{n} a_j x^{j-1}\right)^2 dx = \int_0^1 \frac{d}{da_i} \left(f(x) - \sum_{j=1}^{n} a_j x^{j-1}\right)^2 dx$$

$$= \int_0^1 2\left(f(x) - \sum_{j=1}^{n} a_j x^{j-1}\right) \frac{d}{da_i}\Big(\underbrace{f(x)}_{\text{const with respect to } a_i} - \sum_{j=1}^{n} a_j x^{j-1}\Big) dx$$

the only term that survives is when $j = i$

$$= 2\int_0^1 \left(f(x) - \sum_{j=1}^{n} a_j x^{j-1}\right) x^{i-1} \, dx = 2\int_0^1 \left(f(x) x^{i-1} - \sum_{j=1}^{n} a_j x^{i+j-2}\right) dx = 0$$

Then

$$\int_0^1 \sum_{j=1}^{n} a_j x^{i+j-2} \, dx = \int_0^1 f(x) x^{i-1} \, dx$$

or

$$\sum_{j=1}^{n} a_j \int_0^1 x^{i+j-2} \, dx \approx \int_0^1 f(x) x^{i-1} \, dx$$

$$\int_0^1 x^{i+j-2} \, dx = \frac{1}{i+j-1} x^{i+j-1} \Big|_0^1 = \frac{1}{i+j-1}$$

$$\underbrace{\sum_{j=1}^{n} \frac{1}{i+j-1} a_j}_{\text{matrix vector mult}} \approx \underbrace{\int_0^1 f(x) x^{i-1} \, dx}_{\text{right hand side}}$$

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & & & 1/n+1 \\ \vdots & & & & \\ 1/n & 1/n+1 & \cdots & & 1/2n-1 \end{bmatrix} \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad b = \begin{bmatrix} \int_0^1 f(x)\,dx \\ \int_0^1 f(x)\,x\,dx \\ \vdots \\ \int_0^1 f(x) x^{n-1}\,dx \end{bmatrix}$$

Therefore $Ha = b$ and we need to solve for a.

The condition number of this problem

$$k = \|H^{-1}\| \|H\| .$$

Solving for a in $Ha=b$ involved a large condition number (we will check this). Therefore this is not a good problem to solve on a computer...

A different problem to solve to find the polynomial $p(x)$ would be good. Next time!