Now find a unit vector v such that

$$HA = H\begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ Ha_1 & Ha_2 & \cdots & Ha_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} c & \vdots & & \vdots \\ 0 & Ha_2 & \cdots & Ha_n \\ 0 & & & \\ \vdots & & & \\ 0 & \vdots & & \vdots \end{bmatrix}$$

Idea is find v so that $Ha_1 = ce_1$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$Ha_1 = ce_1 \qquad \text{also} \quad \|v\|_2 = 1$$

$$(I - 2vv^T)a_1 = ce_1$$

$$a_1 - 2vv^Ta_1 = ce_1$$

$$a_1 - 2v(v \cdot a_1) = ce_1$$

$$a_1 - ce_1 = 2v\underbrace{(v \cdot a_1)}_{\text{scalar}}$$

$$v = \frac{a_1 - ce_1}{2(v \cdot a_1)} \qquad \|v\| = \left\| \frac{a_1 - ce_1}{2(v \cdot a_1)} \right\| = 1$$

v is here how do I eliminate that...

$$\frac{1}{|2(v \cdot a_1)|}\|a_1 - ce_1\| = 1$$

$$|2(v \cdot a_1)| = \|a_1 - ce_1\|$$

$$v = \pm \frac{a_1 - ce_1}{\|a_1 - ce_1\|}$$

need actually to find H

$$H = I - 2vv^T$$

the $\sim$ sign immediately cancels when plugging v into the outer product vv^T.

since v multiplies itself you get the same H whether $\pm v$.

But what is c? Plug it in and solve for c

$$a_1 - 2v(v\cdot a_1) = ce_1$$

$$a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\cdot a_1\right) = ce_1$$

$$a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1\cdot a_1 - ce_1\cdot a_1}{\|a_1 - ce_1\|}\right) = ce_1$$

$$\left(a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1\cdot a_1 - ce_1\cdot a_1}{\|a_1 - ce_1\|}\right)\right)\cdot e_1 = ce_1\cdot e_1$$

?

$$\left(a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1\cdot a_1 - ce_1\cdot a_1}{\|a_1 - ce_1\|}\right)\right)\cdot a_1 = ce_1\cdot a_1$$

$$\left(a_1 - 2\frac{a_1 - ce_1}{\|a_1 - ce_1\|}\left(\frac{a_1\cdot a_1 - ce_1\cdot a_1}{\|a_1 - ce_1\|}\right)\right)\cdot e_1 = c$$

$$a_1\cdot e_1 - 2\frac{(e_1\cdot a_1 - c)(a_1\cdot a_1 - ce_1\cdot a_1)}{\|a_1 - ce_1\|^2} = c$$

$$(a_1\cdot e_1 - c) - 2\frac{(e_1\cdot a_1 - c)(a_1\cdot a_1 - ce_1\cdot a_1)}{\|a_1 - ce_1\|^2} = 0$$

$$(a_1\cdot e_1 - c)\left[1 - \frac{2(a_1\cdot a_1 - ce_1\cdot a_1)}{\|a_1 - ce_1\|^2}\right] = 0$$

$$1 - \frac{2(a_1 \cdot a_1 - c e_i \cdot a_1)}{\|a_1 - c e_1\|^2} = \frac{\|a_1 - c e_1\|^2 - 2(\|a_1\|^2 - c e_i \cdot a_1)}{\|a_1 - c e_1\|^2}$$

$$\|a_1 - c e_1\|^2 = (a_1 - c e_1) \cdot (a_1 - c e_1) = a_1 \cdot a_1 - c e_1 \cdot a_1 - c e_1 \cdot a_1 + c^2$$

$$= \|a_1\|^2 - 2 e_1 \cdot a_1 c + c^2$$

$$\frac{\|a_1 - c e_1\|^2 - 2(\|a_1\|^2 - c e_i \cdot a_1)}{\|a_1 - c e_1\|^2} = \frac{\|a_1\|^2 - 2 \cancel{e_1 \cdot a_1} c + c^2 - 2(\|a_1\|^2 - \cancel{c e_i \cdot a_1})}{\|a_1 - c e_1\|^2}$$

$$= \frac{-\|a_1\|^2 + c^2}{\|a_1 - c e_1\|^2}$$

$$\underbrace{(a_1 \cdot e_1 - c)}_{\text{either this is zero}} \underbrace{\left[\frac{-\|a_1\|^2 + c^2}{\|a_1 - c e_1\|^2} \right]}_{\text{or this is zero ...,}} = 0$$

recognize as what I want

$c = a_1 \cdot e_1$ or $c = \pm \|a_1\|$

what? extraneous solution...,

need the value of c to finish finding v

$$v = \frac{a_1 - c e_1}{\|a_1 - c e_1\|}$$

$$a_1 - c e_1 \overset{?}{=} a_1 - a_1 \cdot e_1 e_1 \simeq a_1 - e_1 \cdot a_1 e_1 \quad ?$$

Try with an example...

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{find } H = I - 2vv^T.$$

```
julia> A=[1 2 3; 4 5 6; 7 8 9]
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 7  8  9

julia> e1=[1,0,0]
3-element Vector{Int64}:
 1
 0
 0

julia> A[:,1]
3-element Vector{Int64}:
 1
 4
 7
```

```
julia> c=A[:,1]'*e1
1
```

```
julia> w=A[:,1]-c*e1
3-element Vector{Int64}:
 0
 4
 7
```

```
julia> using LinearAlgebra

julia> v=w/norm(w)
3-element Vector{Float64}:
 0.0
 0.4961389383568338
 0.8682431421244593
```

```
julia> H=I-2*v*v'
3×3 Matrix{Float64}:
  1.0  -0.0        -0.0
 -0.0   0.507692  -0.861538
 -0.0  -0.861538  -0.507692
```

didn't work...

```
julia> H*A
3×3 Matrix{Float64}:
  1.0   2.0       3.0
 -4.0  -4.35385  -4.70769
 -7.0  -8.36923  -9.73846
```

extraneous solution...

The other choice for c . . .

$c = \pm \|a_1\|$

```
julia> c=norm(A[:,1])
8.12403840463596
```

$a_1 - ce_1 \approx a_1 - \|a_1\| e_1$

```
julia> w=A[:,1]-c*e1
3-element Vector{Float64}:
 -7.124038404635961
  4.0
  7.0
```

```
julia> v=w/norm(w)
3-element Vector{Float64}:
 -0.6621587834578151
  0.3717884412453036
  0.6506297721792756
```

```
julia> H=I-2*v*v'
3×3 Matrix{Float64}:
 0.123091   0.492366   0.86164
 0.492366   0.723547  -0.483793
 0.86164   -0.483793   0.153362
```

```
julia> H*A
3×3 Matrix{Float64}:
 8.12404       9.60114   11.0782
 4.44089e-16   0.732119   1.46424
 8.88178e-16   0.531209   1.06242
```

c

zero within rounding.

$c = -\|a_1\|$

```
julia> c=-norm(A[:,1])
       w=A[:,1]-c*e1
       v=w/norm(w)
       H=I-2*v*v'
       H*A
3×3 Matrix{Float64}:
 -8.12404       -9.60114    -11.0782
  8.88178e-16   -0.0859656   -0.171931
  8.88178e-16   -0.90044     -1.80088
```

divide by $\|a_1 - ce_1\|$

c

zero up to rounding

need to make sure $\|a_1 - ce_1\| \neq 0$ and for numerical stability choose the c such that $\|a_1 - ce_1\|$ is largest ...

If $\| a_1 - \|a_1\| e_1 \| > \| a_1 + \|a_1\| e_1 \|$

then $c = \|a_1\|$

otherwise $c = -\|a_1\|$

```
julia> norm(A[:,1]-norm(A[:,1])*e1)
10.758806773556634

julia> norm(A[:,1]+norm(A[:,1])*e1)
12.175716685652304
```

neither is close to 0 but is better.

```
julia> v=w/norm(w)
3-element Vector{Float64}:
 0.7493635602894408
 0.32852275584841295
 0.5749148227347227
```

This v

This is HA ...

```
3×3 Matrix{Float64}:
 -8.12404        -9.60114     -11.0782
  8.88178e-16    -0.0859656    -0.171931
  8.88178e-16    -0.90044      -1.80088
```

Now find a v_2 for this matix so

$$\underbrace{(I - 2v_2v_2^T)}_{2\times2\ \ 2\times2}\begin{bmatrix}-0.0859656 & -0.171931\\ -0.90044 & -1.80088\end{bmatrix} = \begin{bmatrix} c_2 & ? \\ 0 & ? \end{bmatrix}$$

extend v_2 to $\mathbb{R}^3$ as $H_2 = I - 2\begin{bmatrix}0\\v_2\end{bmatrix}\begin{bmatrix}0\\v_2\end{bmatrix}^T = \underbrace{I}_{3\times3} - 2\left[\begin{array}{c|cc}0 & 0 & 0\\ 0 & & \\ 0 & \multicolumn{2}{c}{v_2v_2^T}\end{array}\right]$

$\begin{bmatrix}0\\v_2\end{bmatrix}$