$f(x+h) - f(x)$ exact calculation $|\varepsilon_i| \le \frac{1}{2}\varepsilon_{mach} = \frac{1}{2}2^{-52} = 2^{-53}$

$fl(f(x)) = f(x)(1+\varepsilon_1)$
$fl(f(x+h)) = f(x+h)(1+\varepsilon_2)$

$fl(fl(f(x+h)) - fl(f(x))) = [f(x)(1+\varepsilon_1) - f(x+h)(1+\varepsilon_2)](1+\varepsilon_3)$

$= (f(x) + \varepsilon_1 f(x) - f(x+h) - \varepsilon_2 f(x+h))(1+\varepsilon_3)$

$= f(x) + \varepsilon_1 f(x) - f(x+h) - \varepsilon_2 f(x+h) +$
$\varepsilon_3 f(x) + \varepsilon_1 \varepsilon_3 f(x) - \varepsilon_3 f(x+h) - \varepsilon_2 \varepsilon_3 f(x+h)$

$= \underbrace{f(x) - f(x+h)}_{\text{exact answer}} + (\varepsilon_1 + \varepsilon_3 + \varepsilon_1\varepsilon_3) f(x) - (\varepsilon_2 + \varepsilon_3 + \varepsilon_2\varepsilon_3) f(x+h)$

Error $= fl(fl(f(x+h)) - fl(f(x))) - (f(x) - f(x+h))$

$= (\varepsilon_1 + \varepsilon_3 + \varepsilon_1\varepsilon_3) f(x) - (\varepsilon_2 + \varepsilon_3 + \varepsilon_2\varepsilon_3) f(x+h)$

$|\varepsilon_i| < \frac{1}{2}\varepsilon_{mach}$

sign could be + or minus for each ε_i

In the worst case the signs don't cancel.

$\frac{1}{2}\varepsilon_{mach}$ $\frac{1}{2}\varepsilon_{mach}$ $\frac{1}{4}\varepsilon_{mach}^2$

$|\text{Error}| \le |\varepsilon_1 + \varepsilon_3 + \varepsilon_1\varepsilon_3|\,|f(x)| + |\varepsilon_2 + \varepsilon_3 + \varepsilon_2\varepsilon_3|\,|f(x+h)|$

higher order neglect

$\le \underbrace{(\varepsilon_{mach} + \frac{1}{4}\varepsilon_{mach}^2)}_{\delta} |f(x)| + \underbrace{(\varepsilon_{mach} + \frac{1}{4}\varepsilon_{mach}^2)}_{\delta} |f(x+h)|$

$\delta \approx \varepsilon_{mach}$

$$|\text{error}| \leq \delta\left(|f(x)| + |f(x+h)|\right)$$

$$\overbrace{\left| \frac{fl(fl(f(x+h)) - fl(f(x)))}{h} - \frac{f(x+h)-f(x)}{h} \right|}^{\text{generated error}} \approx \frac{\delta\left(|f(x)| + |f(x+h)|\right)}{h}$$

mean value theorem $f(x+h) - f(x) = f'(\xi)h$ for some ξ between x and $x+h$

$$\underbrace{\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right|}_{\text{truncation error}} \approx \left| \frac{f'(\xi)\cancel{h}}{\cancel{h}} - f'(x) \right| = |f'(\xi) - f'(x)|$$

mean value theorem $f'(\xi) - f'(x) = f''(\eta)(\xi - x)$ for some η between x and ξ.

$$\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right| = |f''(\eta)||\xi - x| \leq |f''(\eta)|h$$

$$|\text{total error}| \leq \frac{\delta\left(|f(x)| + |f(x+h)|\right)}{h} + |f''(\eta)|h$$

if h is too small this term is large

if h is too large this term is large.

Try to minimize the error...

depends on h...

also depends on h...

$$\min\left\{ \frac{\delta\left(|f(x)| + |f(x+h)|\right)}{h} + |f''(\eta)|h \ : \ h > 0 \right\}$$

Unless you know exactly what f is can't exactly minimize...

$C_1 \approx |f(x)| + |f(x+h)|$ $\qquad$ $C_2 \approx |f''(\eta)|$

h is small so not much dependency (hopefully) or take upper bound

alternatively $|f(x)| + |f(x+h)| \leq C_1$ for h sufficiently small.

Consider simpler upper bound

$$e(h) = \frac{\delta C_1}{h} + C_2 h$$

This can be minimized

$$e'(h) = -\frac{\delta C_1}{h^2} + C_2 = 0$$

So h is proportional to $\sqrt{\delta}$

$$h^2 = \delta \frac{C_1}{C_2} \quad \text{or} \quad h = \sqrt{\delta}\underbrace{\sqrt{\frac{C_1}{C_2}}}_{\text{const}}$$

$\delta \approx \text{const}\sqrt{\varepsilon_{mach}}$

compared to ε_{mach} if f and f'' are not really big or small then

heuristically the optimal value of h

$$h = \sqrt{\varepsilon_{mach}} = \sqrt{2^{-53}} = 2^{-53/2}$$

```
julia> f(x)=sin(x)
       df(x)=cos(x)
       dfh(x,h)=(f(x+h)-f(x))/h
dfh (generic function with 2 methods)

julia> xs=0:0.05:2*pi
0.0:0.05:6.25

julia> using LinearAlgebra

julia> norm(dfh(x,0.5)-df(x) for x in xs)
1.9680568313855789
```

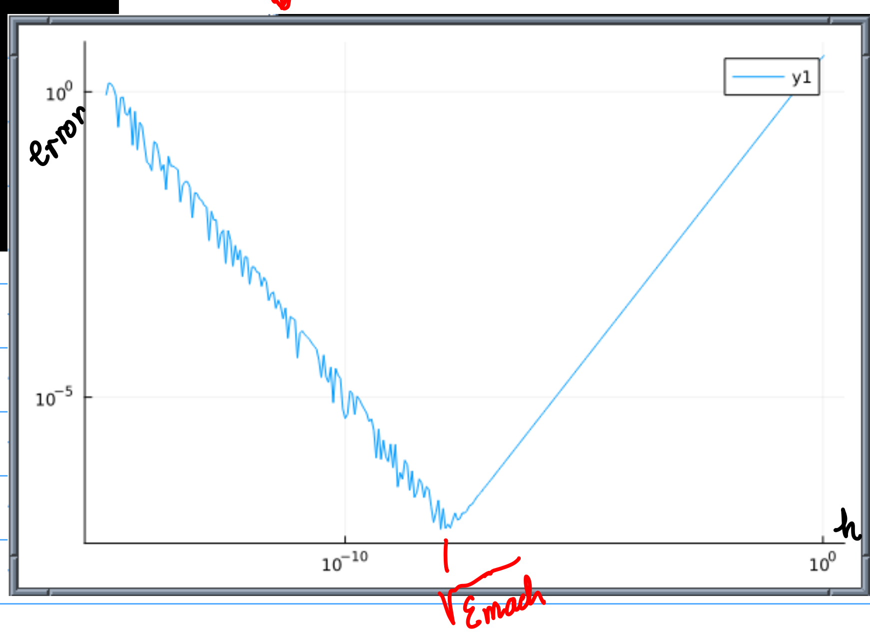
```
julia> eh(h)=norm(dfh(x,h)-df(x) for x in xs)
eh (generic function with 1 method)

julia> eh(0.5)
1.9680568313855789
```

$\sqrt{\varepsilon_{mach}}$.

```
julia> eh(sqrt(2.0^(-52)))
6.837260857196583e-8
```

```
julia> hs=10.0.^(-15:0.05:0)
301-element Vector{Float64}:
 1.0e-15
 1.1220184543019653e-15
 1.2589254117941663e-15
 1.4125375446227554e-15
 1.584893192461111e-15
 1.7782794100389227e-15
 1.995262314968883e-15
 2.2387211385683378e-15
 2.511886431509582e-15
 2.818382931264449e-15
 ⋮
 0.3981071705534972
 0.44668359215096315
 0.5011872336272722
 0.5623413251903491
 0.6309573444801932
 0.7079457843841379
 0.7943282347242815
 0.8912509381337456
 1.0
```

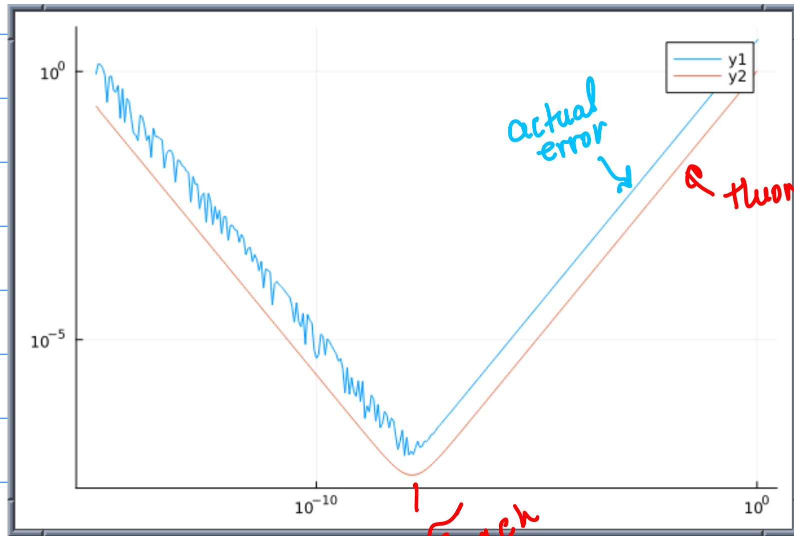
```
julia> plot(eh,hs,xscale=:log10,yscale=:log10)
```



```
julia> et(h)=2.0^(-52)/h+h
et (generic function with 1 method)
```

← Theoretical curve with $c_1 = 1$ and $c_2 = 1$.

```
julia> plot!(et,hs,xscale=:log10,yscale=:log10)
```



Are there better approximations for the derivative?

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

← difference quotient

Calculus definition of $f'(x)$ is the limit of the difference quotient.

Central difference approximation.

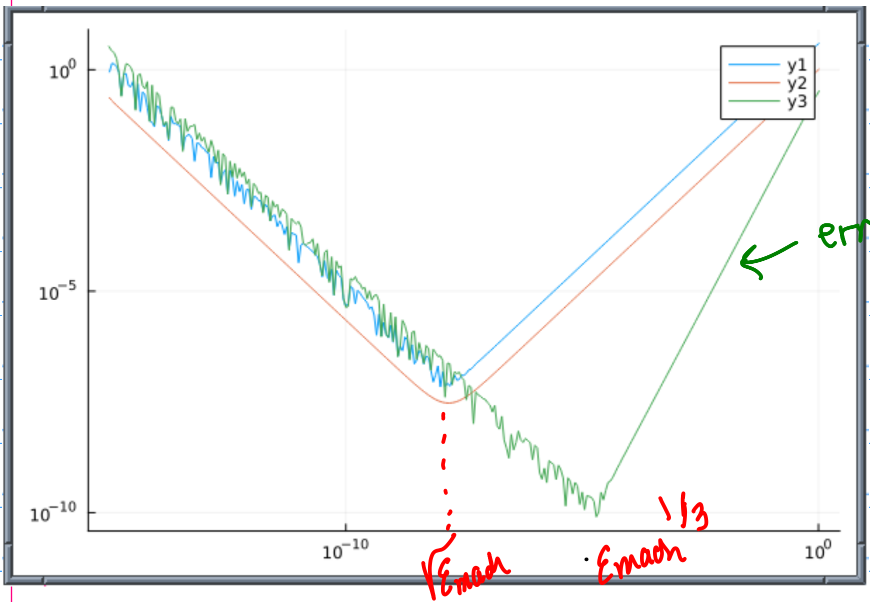
x is halfway between $x + \frac{h}{2}$ and $x - \frac{h}{2}$

$$f'(x) \approx \frac{f(x+h/2) - f(x-h/2)}{h}$$

central difference approx

```
julia> dfc(x,h)=(f(x+h/2)-f(x-h/2))/h
dfc (generic function with 1 method)

julia> ec(h)=norm(dfc(x,h)-df(x) for x in xs)
ec (generic function with 1 method)

julia> plot!(ec,hs,xscale=:log10,yscale=:log10)
```

error in central difference over the interval $[0,2\pi]$



error in central difference decreases faster.

Next time general way of figuring out a good value of h...