Two approximations of $f'(x)$

$$f'(x) \approx \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{forward difference quotient}} \qquad \text{also} \qquad f'(x) \approx \underbrace{\frac{f(x+h/2)-f(x-h/2)}{h}}_{\text{central difference}}$$

Analyse the truncation error:

$$\tau_h = f'(x) - \frac{f(x+h/2)-f(x-h/2)}{h}$$

Taylor's Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ with $n+1$ derivatives. Then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + R_n$$

where $R_n = \dfrac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$ for some ξ between x and $x+h$.

This cancellation came from the symmetry in the formula.

$$f(x+h/2) = \cancel{f(x)} + \frac{h}{2}f'(x) + \boxed{\cancel{\frac{(h/2)^2}{2}} f''(x)} + \frac{(h/2)^3}{6} f^{(3)}(\xi_1) \quad \text{for some } \xi_1 \text{ between } x \text{ and } x+h/2$$

$$f(x-h/2) = \cancel{f(x)} - \frac{h}{2}f'(x) + \boxed{\cancel{\frac{(h/2)^2}{2}} f''(x)} - \frac{(h/2)^3}{6} f^{(3)}(\xi_2) \quad \text{for some } \xi_2 \text{ between } x \text{ and } x-h/2$$

$$f(x+h/2) - f(x-h/2) = hf'(x) + \frac{(h/2)^3}{6}\left(f^{(3)}(\xi_1) + f^{(3)}(\xi_2)\right)$$

Thus,

$$\frac{f(x+h/2)-f(x-h/2)}{h} = f'(x) + \frac{h^2}{48}\left(f^{(3)}(\xi_1) + f^{(3)}(\xi_2)\right)$$

almost constant

$$\tau_h = f'(x) - \frac{f(x+h/2)-f(x-h/2)}{h} = -\frac{h^2}{48}\left(f^{(3)}(\xi_1) + f^{(3)}(\xi_2)\right)$$

generated error

$$\left| \frac{\underbrace{fl(fl(f(x+h/2)) - fl(f(x-h/2)))}_{}}{h} - \frac{\underbrace{f(x+h/2) - f(x-h/2)}_{}}{h} \right| \leq \frac{\delta \underbrace{\left(|f(x-h/2)| + |f(x+h/2)|\right)}_{}}{h}$$

truncation error

$$|\tau_h| = h^2 \left| \frac{1}{48}\left(f^{(3)}(\xi_1) + f^{(3)}(\xi_2)\right) \right|$$

$$|\text{Total error}| \leq \underbrace{\frac{\delta C_1}{h} + C_2 h^2}_{} = e(h)$$

↑ minimize the upper bound.

$$e'(h) = -\frac{\delta C_1}{h^2} + 2C_2 h = 0$$

$\delta \approx \varepsilon_{mach}$

$$\delta C_1 = 2C_2 h^3$$

$$h^3 = \frac{\delta C_1}{2C_2}, \quad h \approx \varepsilon_{mach}^{1/3} \cdot \text{const}$$

$$\text{const} = \left(\frac{C_1}{2C_2}\right)^{1/3}$$

General idea... Finite difference approx

$$f'(x) \approx \frac{1}{h} \sum_{k=-p}^{q} a_k f(x+kh)$$

centered difference could also be written

$$f'(x) \approx \frac{f(x+h/2) - f(x-h/2)}{h} = \frac{f(x+\eta) - f(x-\eta)}{2\eta}$$

$$\eta = h/2 \qquad h = 2\eta$$

In terms of finite difference we can take

$p=-1$, $q=1$ $\quad a_{-1} = \frac{1}{2}$, $a_0 = 0$, $a_1 = \frac{1}{2}$ to get central difference.

$$f'(x) \approx \frac{1}{h} \sum_{k=-p}^{q} a_k f(x+kh)$$

$$k=-2 \quad a_{-2} f(x-2h) = \left(f(x) - 2hf'(x) + \frac{(2h)^2}{2!} f''(x) - \frac{(2h)^3}{3!} f^{(3)}(x) + \frac{(2h)^4 f^{(4)}(x)}{4!} + \dots\right) a_{-2}$$

$$k=-1 \quad a_{-1} f(x-h) = \left(f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4 f^{(4)}(x)}{4!} + \dots\right) a_{-1}$$

$$k=0 \quad a_0 f(x) = f(x)\, a_0$$

$$k=1 \quad a_1 f(x+h) = \left(f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4 f^{(4)}(x)}{4!} + \dots\right) a_1$$

$$k=2 \quad a_2 f(x+2h) = \left(f(x) + 2hf'(x) + \frac{(2h)^2}{2!} f''(x) + \frac{(2h)^3}{3!} f^{(3)}(x) + \frac{(2h)^4 f^{(4)}(x)}{4!} + \dots\right) a_2$$

$$\sum_{k=-2}^{2} a_k f(x+kh) = \overbrace{\sum_{k=-2}^{2} a_k}^{=0} f(x) + \overbrace{\left(-2a_{-2} - a_{-1} + a_1 + 2a_2\right)}^{=1} h f'(x)$$

$$+ \overbrace{\left(2a_{-2} + \tfrac{1}{2} a_{-1} + \tfrac{1}{2} a_1 + 2a_2\right)}^{=0} h^2 f''(x)$$

$$+ \overbrace{\left(-\tfrac{4}{3} a_{-2} - \tfrac{1}{6} a_{-1} + \tfrac{1}{6} a_1 + \tfrac{4}{3} a_2\right)}^{=0} h^3 f^{(3)}(x)$$

$$+ \left(\cdot \; - \; - \; - \right)$$

dependending on the derivative you want to approximate set the coefficience in terms of the a_k's so all but one of them vanish.

In the end we have an approximation of, for example f', so that

$$\tau_h = \left| f'(x) - \frac{1}{h} \sum_{k=-p}^{q} a_k f(x+kh) \right| = \boxed{\mathcal{O}(h^m)}$$

means $\left|\frac{\tau_h}{h^m}\right|$ is bounded by a constant as $h \to 0$.

Generated error.

$$\left| \frac{1}{h} fl\left(\sum_{k=-p}^{q} fl\left(a_k\, fl\left(f(x+kh)\right)\right)\right) - \frac{1}{h} \sum_{k=-p}^{q} a_k f(x+kh) \right| \lesssim \frac{\varepsilon_{mach} C_1}{h}$$

minimize upper bound of Total error

$$|\text{Total error}| \lesssim \frac{\varepsilon_{mach} C_1}{h} + C_2 h^m = e(h)$$

$$e'(h) = -\frac{\varepsilon_{mach} C_1}{h^2} + m C_2 h^{m-1} = 0$$

$$m C_2 \cdot h^{m+1} = \varepsilon_{mach} C_1 \qquad h = \varepsilon_{mach}^{1/m+1} \cdot \text{const}$$

$\boxed{\text{Given an order } O(h^m) \text{ approximation in general taking } h \approx \varepsilon_{mach}^{1/m+1} \text{ is reasonable...}}$