Numerical Quadrature — approximating integrals $\int_a^b f(x)\,dx$

Two approximations of $f'(x)$

$$\boxed{f'(x) \approx \underbrace{\frac{f(x+h)-f(x)}{h}}_{\text{forward difference quotient}}}$$

definition of derivative

also

$$\boxed{f'(x) \approx \underbrace{\frac{f(x+h/2)-f(x-h/2)}{h}}_{\text{central difference}}}$$

worked better than definition...

These definitions that are convenient for theoretical mathematics may not yield good numerical methods...

Definition of integral... Riemann sum...

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \overbrace{f(x_k^*)\underbrace{\Delta x}_{h}}^{\text{areas of rectanges all added up}}$$

where $\underbrace{\Delta x}_{h} = \frac{b-a}{n}$ and $x_k^* \in [x_{k-1}, x_k]$ and $x_k = a + k\underbrace{\Delta x}_{h}$

left or right end poin

Just take n really big and for definiteness

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \boxed{f(x_{k-1})\,h}$$

left Riemann sum

$$\approx \sum_{k=1}^{n} \boxed{f(x_k)\,h}$$

right Riemann sum.

These formula are not symmetric...

Think about breaking the area into pieces

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \boxed{\int_{x_{k-1}}^{x_k} f(x)\,dx}$$

Try to make a more symmetric formula.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left(\sum_{k=1}^{n} f(x_{k-1})\,h + \sum_{k=1}^{n} f(x_k)\,h\right)$$

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \frac{f(x_{k-1})+f(x_k)}{2}\,h$$

Trapezoid rule..,

For each piece we have this approximation

$$\boxed{\int_{x_{k-1}}^{x_k} f(x)\,dx} \approx \frac{f(x_{k-1})+f(x_k)}{2}\,h$$

lengths of the parallel sides

distance between parallel sides

$h = x_k - x_{k-1}$

since $x_k = a + kh$

Another symmetric formula... for n large

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) h$$

midpoint rule

Questions: How good (or bad) are these approximations...

Idea 1: Try it out on a computer and see for some test cases how things.

Idea 2: Do some calculus to estimate the expected error.

Two ideas work together to answer the question...

put inside.

$$e_k = \left| \frac{f(x_{k-1}) + f(x_k)}{2} h - \int_{x_{k-1}}^{x_k} f(x)\,dx \right|$$

length of this interval $[x_{k-1}, x_k]$ is h.

Compare terms and cancel them...

$$= \left| \int_{x_{k-1}}^{x_k} \left(\frac{f(x_{k-1}) + f(x_k)}{2} - f(x) \right) dx \right|$$

Use Taylor theorem to estimate the error...

$$f(x_{k-1}) = f(x) + (x_{k-1} - x) f'(x) + \frac{(x_{k-1} - x)^2}{2} f''(x) + R_2$$

To avoid all the subscripts, change the variables

$$\int_{x_{k-1}}^{x_k} \left(\frac{f(x_{k-1}) + f(x_k)}{2} - f(x) \right) dx = \int_0^h \left(\frac{g(0) + g(h)}{2} - g(t) \right) dt$$

$x = t + x_{k-1}$

$g(t) = f(t + x_{k-1})$

Now estimate the size of

$$e_k = \int_0^h \left(\frac{g(0)+g(h)}{2} - g(t) \right) dt$$

Using Taylor's theorem

remainder term

$$g(0) = g(t) - t g'(t) + \frac{t^2}{2} g''(\xi_1(t))$$ where $\xi_1(t)$ is between t and 0

$$g(h) = g(t) + (h-t) g'(t) + \frac{(h-t)^2}{2} g''(\xi_2(t))$$ and $\xi_2(t)$ is between t and h,

$$\frac{g(0)+g(h)}{2} = g(t) + \frac{(h-2t)}{2} g'(t) + \frac{t^2 g''(\xi_1(t)) + (h-t)^2 g''(\xi_2(t))}{4}$$

$$e_k = \int_0^h \left(\frac{(h-2t)}{2} g'(t) + \frac{t^2 g''(\xi_1(t)) + (h-t)^2 g''(\xi_2(t))}{4} \right) dt$$

let B be a bound so $|g''(t)| \le B$ for all $t \in [a,b]$

for example g'' is continuous ..
then take B to be the maximum.

Integrate by parts

cancel ... zero at endpoints ... cancellation due to the symmetry

$$\int_0^h \frac{(h-2t)}{2} g'(t)\, dt = g'(t) \frac{(ht-t^2)}{2} \Big|_0^h - \int_0^h \frac{ht-t^2}{2} g''(t)\, dt$$

$u = g'(t) \qquad du = g''(t) dt$

$dv = \frac{(h-2t)}{2} dt \qquad v = \frac{(ht-t^2)}{2}$

Finish Monday
Lab next time