$$e_k = \int_0^h \left(\frac{g(0)+g(h)}{2} - g(t) \right) dt$$

using Taylor's theorem

remainder ter...

$$g(0) = g(t) - t g'(t) + \frac{t^2}{2} g''(\xi_1(t))$$ where $\xi_1(t)$ is between t and 0

$$g(h) = g(t) + (h-t) g'(t) + \frac{(h-t)^2}{2} g''(\xi_2(t))$$ and $\xi_2(t)$ is between t and h,

$$\frac{g(0)+g(h)}{2} = g(t) + \frac{(h-2t)}{2} g'(t) + \frac{t^2 g''(\xi_1(t)) + (h-t)^2 g''(\xi_2(t))}{4}$$

$$e_k = \int_0^h \left(\frac{(h-2t)}{2} g'(t) + \frac{t^2 g''(\xi_1(t)) + (h-t)^2 g''(\xi_2(t))}{4} \right) dt$$

let B be a bound so $|g''(t)| \le B$ for all $t \in [a,b]$

for example g'' is continuous ..

then take B to be the maximum.

cancel ...

symm...

$$\int_0^h \frac{(h-2t)}{2} g'(t)\, dt = g'(t) \frac{(ht-t^2)}{2} \Big|_0^h - \int_0^h \frac{ht-t^2}{2} g''(t)\, dt$$

$u = g'(t)$ $\quad du = g''(t)\,dt$

$dv = \frac{(h-2t)}{2} dt$ $\quad v = \frac{(ht-t^2)}{2}$

Finish Monday

$$|e_k| = \left| \int_0^h \left[-\frac{ht-t^2}{2} g''(t) + \frac{t^2 g''(\xi_1(t)) + (h-t)^2 g''(\xi_2(t))}{4} \right] dt \right|$$

$$\leq B \int_0^h \left(\frac{ht-t^2}{2} + \underbrace{\frac{t^2}{4}}_{t^2/2} + \frac{t^2}{4} \right) dt = B \left(\frac{h\frac{1}{2}t^2 - \frac{1}{3}t^3}{2} + \frac{t^3}{6} \right) \Bigg|_{t=0}^{h}$$

$$= B \left(\frac{3h^3 - 2h^3}{12} + \frac{h^3}{6} \right) = Bh^3 \left(\frac{1}{12} + \frac{2}{12} \right) = \frac{B}{4} h^3$$

Therefore

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2} \right) h$$

recall

$$\underbrace{\Delta x}_{h} = \frac{b-a}{n}$$

$$n = \frac{b-a}{h}$$

Total error after adding up all the approximation.

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2} \right) \right| \leq \sum_{k=1}^{n} \frac{B}{4} h^3 = \frac{nB}{4} h^3 = \frac{(b-a)B}{4} h^2 = O(h^2)$$

bounded by a constant times h^2 as $h \to 0$.

Test this: Think of a test problem...

$$\int_1^5 \ln x\,dx = x\ln x - x \Big|_1^5 = 5\ln 5 - 5 - (1\ln 1 - 1) = 5\ln 5 - 4$$

$$\frac{d\, x\ln x}{dx} = \ln x - \frac{x}{x} = \ln x - \frac{d}{dx} x$$

other side

$$h = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n} \qquad x_k = a + kh = 1 + k\frac{4}{n}$$

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2} h$$

$$\int_1^5 \ln x\,dx \qquad \sum_{k=1}^{n} \frac{\ln x_{k-1} + \ln x_k}{2} \cdot \frac{4}{n}$$

```
julia> function trapquad(f,a,b,n)
           h=(b-a)/n
           x(k)=a+k*h
           return sum((f(x(k-1))+f(x(k)))/2*h for k=1:n)
       end
trapquad (generic function with 1 method)
```

```
julia> trapquad(log,1.0,5.0,5)
4.005501618446435
```

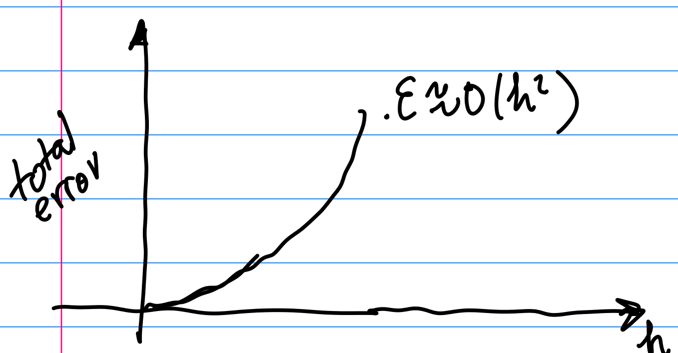
```
julia> A=5*log(5)-4
4.047189562170502
```

```
julia> trapquad(log,1.0,5.0,10)
4.036590511575611
```

← expected the error to decrease faster or not?

```
julia> trapquad(log,1.0,5.0,100)
4.047082902554811
```

let's draw a graph..



logscale to see better

$$\varepsilon = M h^2$$

$$\log \varepsilon = \log M + 2 \log h$$

looking for a line with slope 2 in log-log coordinates..

```
julia> ns=2.0.^(1:16)
16-element Vector{Float64}:
     2.0
     4.0
     8.0
    16.0
    32.0
    64.0
   128.0
   256.0
   512.0
  1024.0
  2048.0
  4096.0
  8192.0
 16384.0
 32768.0
 65536.0
```

```
julia> hs=(5-1)./ns
16-element Vector{Float64}:
 2.0
 1.0
 0.5
 0.25
 0.125
 0.0625
 0.03125
 0.015625
 0.0078125
 0.00390625
 0.001953125
 0.0009765625
 0.00048828125
 0.000244140625
 0.0001220703125
 6.103515625e-5
```

```
julia> E(n)=abs(trapquad(log,1.0,5.0,n)-A)
E (generic function with 1 method)
```

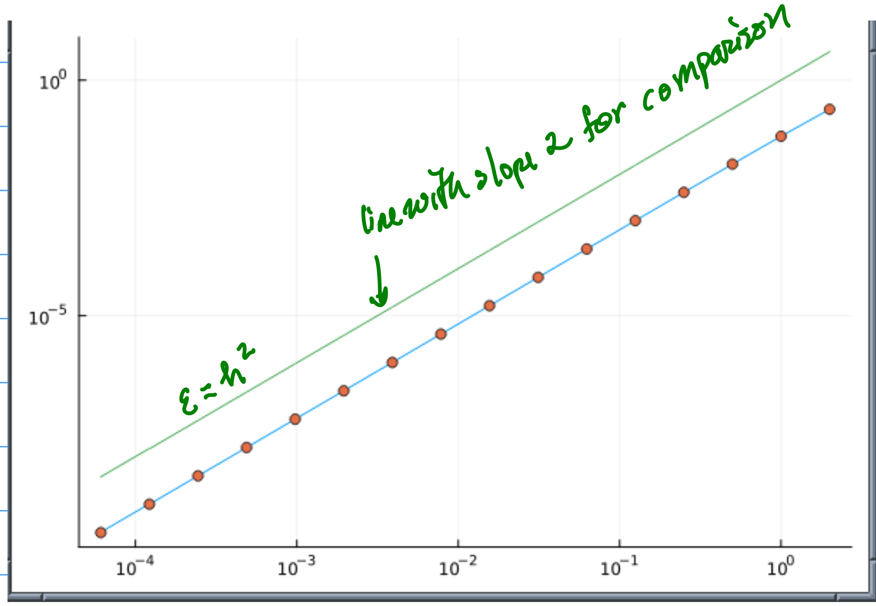
```
julia> Es=E.(ns)
16-element Vector{Float64}:
 0.240527072400182
 0.06441677560550563
 0.016505066261022883
 0.0041560881029418795
 0.0010409969154583365
 0.00026037466737793835
 6.510153950323172e-5
 1.6275877433358232e-5
 4.069001501531015e-6
 1.0172519591122864e-6
 2.5431311012624747e-7
 6.357828041814173e-8
 1.5894569216357013e-8
 3.973627649145328e-9
 9.934000289035794e-10
 2.483577787870672e-10
```

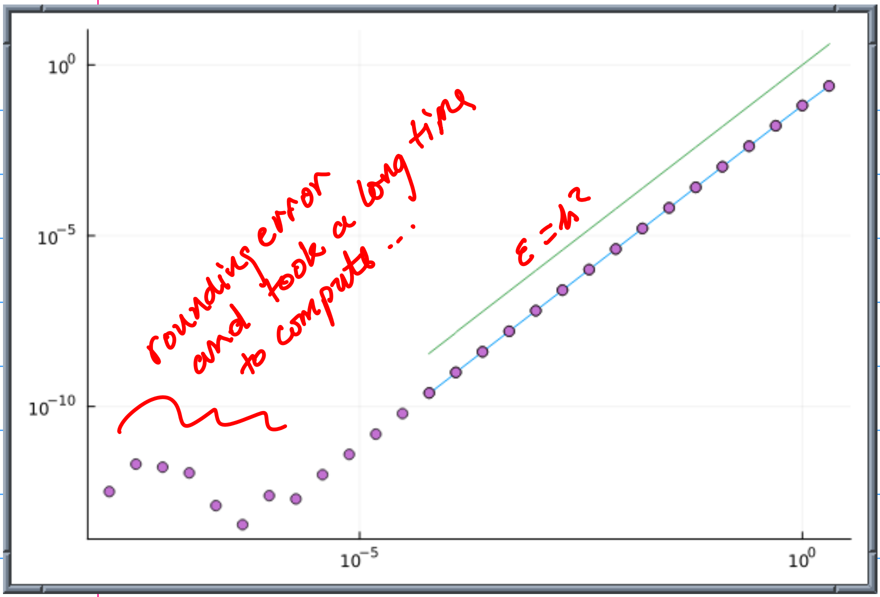
```
julia> using Plots

julia> plot(hs,Es,xscale=:log10,yscale=:log10,legend=false)

julia> scatter!(hs,Es,xscale=:log10,yscale=:log10,legend=false)

julia> plot!(h->h^2,hs)
```





Idea... Extrapolation based on the rate of convergence

$$\varepsilon(h) \approx c h^2 + \dots$$

I didn't compute the constant... but it turns out we don't need to

$$\varepsilon\left(\frac{h}{2}\right) \approx c\left(\frac{h}{2}\right)^2 + \dots$$

$$= \frac{c}{4} h^2 + \dots$$

$$4\,\varepsilon\left(\frac{h}{2}\right) = c h^2 + \dots$$

$$\varepsilon(h) - 4\,\varepsilon\left(\frac{h}{2}\right) = \dots$$

only higher order terms left

Now solve for the integral...

Next time (5.6.10) in the book...