$$\mathcal{E}(h) \approx ch^2 + \cdots$$

higher order term

$$\mathcal{E}\left(\frac{h}{2}\right) \approx c\left(\frac{h}{2}\right)^2 + \cdots$$

$$= \frac{c}{4}h^2 + \cdots$$

$$4\,\mathcal{E}\left(\frac{h}{2}\right) = ch^2 + \cdots$$

$$\mathcal{E}(h) - 4\,\mathcal{E}\left(\frac{h}{2}\right) = \cdots$$

are all that's left after canceling the ch^2 term in the error...

the next term in these higher order terms is ch^4 because of the symmetry we have the h^3 term is already gone.

```
E(n)=abs(trapquad(log,1.0,5.0,n)-A)
```

trapquad: numerical approximation; log: f; 1.0: a; 5.0: b; n: n; A: "exact" answer

Theoretically

$$\mathcal{E}_T(h) = T_h(f,a,b) - \int_a^b f(x)\,dx$$

$$\mathcal{E}_T\left(\frac{h}{2}\right) = T_{\frac{h}{2}}(f,a,b) - \int_a^b f(x)\,dx$$

$$T_h(f,a,b) = \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}\,h$$

where $x_k = a + kh$ and $h = \frac{b-a}{n}$

smaller error

$$\mathcal{E}(h) - 4\,\mathcal{E}\left(\tfrac{h}{2}\right) \approx 0$$

$$T_h(f,a,b) - \int_a^b f(x)\,dx - 4\left(T_{\frac{h}{2}}(f,a,b) - \int_a^b f(x)\,dx\right) \approx 0$$

$$3\int_a^b f(x)\,dx + T_h(f,a,b) - 4T_{h/2}(f,a,b) \approx 0$$

combined two copies of Trapezoid rule to cancel out the error I was seeing ... so that should be less error in this new formula

$$\int_a^b f(x)\,dx \approx \boxed{\frac{4T_{h/2}(f,a,b) - T_h(f,a,b)}{3}}$$

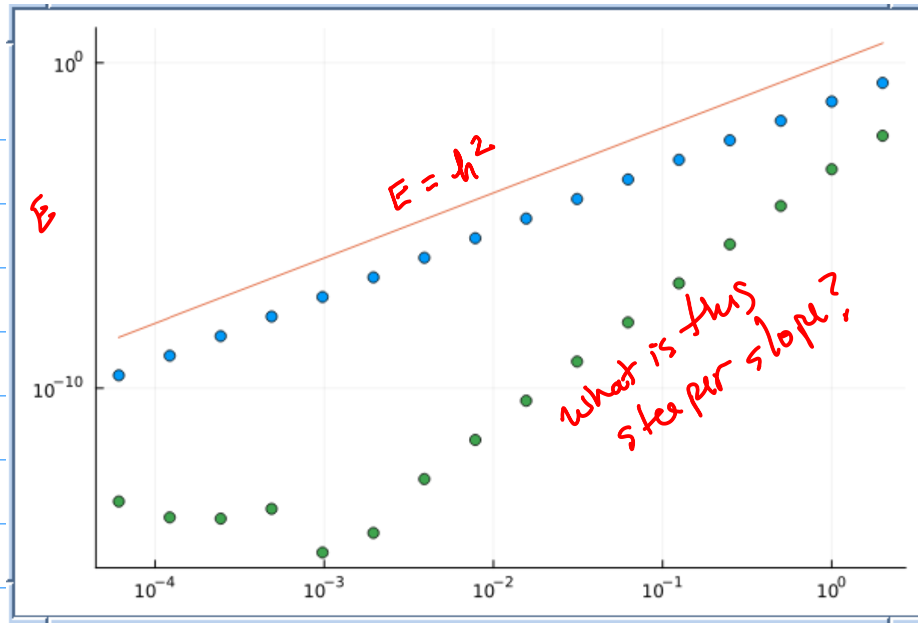
$$S_{h/2}(f,a,b) = \frac{4T_{h/2}(f,a,b) - T_h(f,a,b)}{3}$$

$$\mathcal{E}_S(h) = S_{h/2}(f,a,b) - \int_a^b f(x)\,dx$$

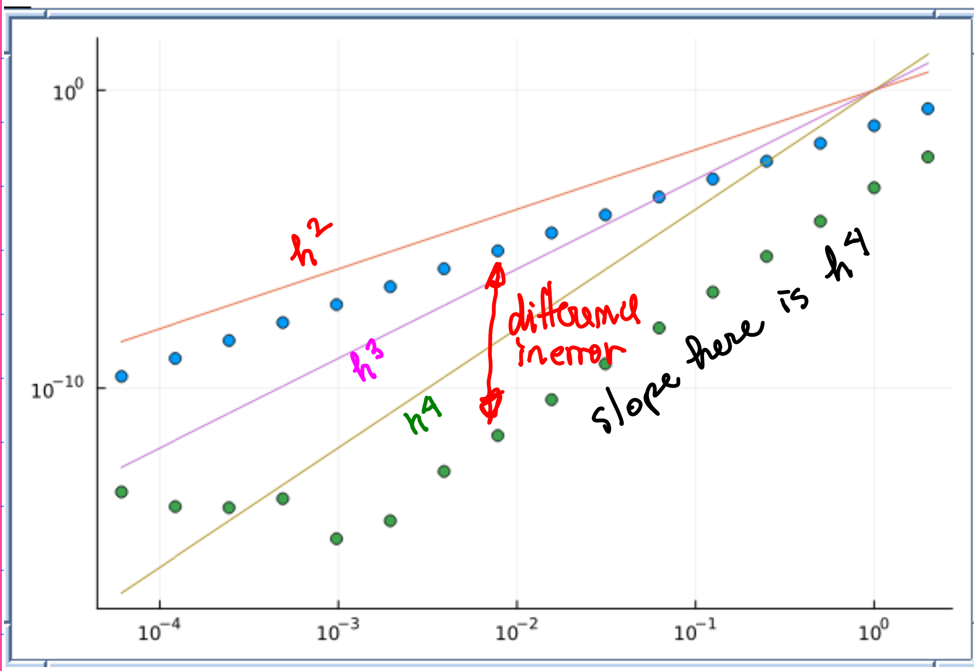
```
julia> function simpquad(f,a,b,n)
           return (4*trapquad(f,a,b,2*n)-trapquad(f,a,b,n))/3
       end
simpquad (generic function with 1 method)

julia> ES(n)=abs(simpquad(log,1.0,5.0,n)-A)
ES (generic function with 1 method)
```

```
julia> ESs=ES.(ns)
16-element Vector{Float64}:
 0.005713343340613797
 0.0005344964795286344
 3.976205024880386e-5
 2.633186297451573e-6
 1.672513514350271 2e-7
 1.049687892162865e-8
 6.567431043436045e-10
 4.105604745063829e-11
 2.562394740834861 3e-12
 1.607602939657226 7e-13
 3.552713678800501e-15
 8.881784197001252e-16
 1.953992523340275 5e-14
 9.769962616701378e-15
 1.065814103640150 3e-14
 3.286260152890463 4e-14
```

smaller errors




```
julia> scatter!(hs,ESs,xscale=:log10,yscale=:log10,legend=false)
```



$$\mathcal{E}_S(h) = S_h(f,a,b) - \int_a^b f(x)\,dx \approx O(h^4)$$

numerically looks like this...

$$\mathcal{E}_S(h) = c h^4 + \dots \qquad \mathcal{E}_S\left(\frac{h}{2}\right) = c\left(\frac{h}{2}\right)^4 = \frac{c}{16} h^4 + \dots$$

Cancel the errors $\mathcal{E}_S(h) - 16\,\mathcal{E}_S\left(\frac{h}{2}\right) = \dots$ cancels the next term...

$$\mathcal{E}_S(h) \approx S_h(f,a,b) - \cancel{\int_a^b f(x)\,dx}$$

$$\mathcal{E}_T(h) \approx T_h(f,a,b) - \cancel{\int_a^b f(x)\,dx}$$

$$\mathcal{E}_T(h) - \mathcal{E}_S(h) \approx S_h(f,a,b) - T_h(f,a,b)$$

Can compute the difference in the errors between the two approximations without knowing the exact solution. The difference between an accurate approximation and a less accurate approximation is related to the error in the less accurate approximation...

How good the good approximation is "has" to be better than how bad the bad approx. was.

- See the code and discussion in chapter 5.7 for an example how to change h to achieve a desired quality in the approximation...

Initial Value Problems (Chapter 6)

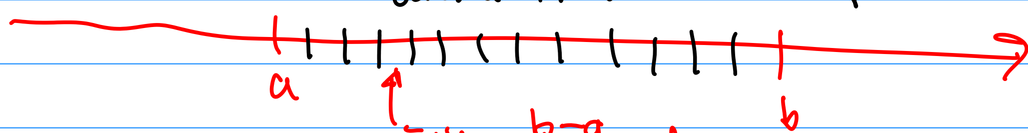
$$u'(t) = f(t, u(t)) \qquad u(a) = u_0$$

ordinary diff. equation

initial condition

Idea approximate the solution forward in time on the interval $[a,b]$.

divide time into n steps



width $= \frac{b-a}{n} = h$

$$t_i = a + ih$$

$$\int_{t_i}^{t_{i+1}} u'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, u(t))\,dt$$

use fund. Theorem of Calculus

$$u(t_{i+1}) - u(t_i) = \int_{t_i}^{t_{i+1}} f(t, u(t))\,dt$$

now approximate this integral...

use left end point

$$u(t_{i+1}) - u(t_i) \approx f(t_i, u(t_i))\,(t_{i+1} - t_i) = h f(t_i, u(t_i))$$

Next time approximate with right endpoint

trapezoid rule

mid point

etc. ,

See a pattern all approximation fit into. This is called Butcher Tableau. The related methods are called Runge-Kutta methods and there are lots of them that satisfy various kinds of optimization conditions..,