Idea is Solve $Ax = b$ for x. Approximation of x is x^*.

$$e = x^* - x \qquad \varepsilon = \|x^* - x\| \qquad \delta = \frac{\|x^* - x\|}{\|x\|}$$

I want to know $\frac{\|x^* - x\|}{\|x\|}$ ← don't know x so can't compute this

residual error... plug x^* into $Ax = b$ and see how wrong it is

Let $b^* = Ax^*$ then $b^* \approx b$ residual error $\|b^* - b\|$

or relative residual error $\frac{\|b^* - b\|}{\|b\|}$

Vector and Matrix norms:

$$\|x^* - x\| = \|e\| = \sqrt{e_1^2 + e_2^2 + \cdots + e_n^2} \quad \text{where } e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \in \mathbb{R}^n$$

Other norms.

$$\|x^* - x\|_p = \|e\|_p = \sqrt[p]{|e_1|^p + |e_2|^p + \cdots + |e_n|^2} \quad \text{where } p \geq 1.$$

Vector Norm properties:

- positivity $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$
  and $\|x\| = 0$ only when $x = 0$ (definiteness).
- homogeneity $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$
- triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ $x, y \in \mathbb{R}^n$

Matrix norms:

- positivity $\|A\| \geq 0$ for all $A \in \mathbb{R}^{m \times n}$ and $\|A\| = 0$ only when $A = 0$ (definiteness).
- homogeneity $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}^{m \times n}$
- triangle inequality $\|A+B\| \leq \|A\| + \|B\|$, $A, B \in \mathbb{R}^{m \times n}$
- sub?: $\|AB\| \leq \|A\| \|B\|$ for $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$
- compatibility $\|Ax\| \leq \|A\| \|x\|$ for $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$

Induced matrix norm: Given a vector norm find a matrix norm that is compatible with that vector norm in an efficient way.

$$\|A\|_p = \max \left\{ \frac{\|Ax\|_p}{\|x\|_p} \text{ for all } x \neq 0 \text{ and } x \in \mathbb{R}^n \right\}.$$

what is this for $p = 1$?

$\|Ax\| \leq \|A\| \|x\|$ so $\frac{\|Ax\|}{\|x\|} \leq \|A\|$

Examples: $\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^2}$ with $p = 1$.

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$Ax =$ what is this? $Ax \in \mathbb{R}^m$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

ith component of the vector

$$[Ax]_i = \sum_{j=1}^{n} a_{ij} x_j$$

switch order of sums

$$\|Ax\|_1 = \sum_{i=1}^{m} |[Ax]_i| = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} x_j|$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}||x_j| = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}| \right) |x_j|$$

sum of the abs values of the jth column

$$M = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{ij}| \quad \text{then} \quad \sum_{i=1}^{m} |a_{ij}| \leq M \text{ for all } j$$

Thus

$$\|Ax\|_1 \leq \sum_{j=1}^{n} M |x_j| = M \sum_{j=1}^{n} |x_j| = M \|x\|_1$$

so $$\frac{\|Ax\|_1}{\|x\|_1} \leq M = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

Want

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq M \quad \text{so} \quad \|A\|_1 \leq M.$$

want to show equal...

Recall

$$M = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{ij}| \quad \text{let } k \text{ be fixed so that } M = \sum_{i=1}^{m} |a_{ik}|$$

the j that makes this maximum, we'll call k.

Let e_k be the standard basis vector with 1 is the k^{th} slot and zeros everywhere else

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots \quad e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow k\text{th entry.}$$

$$[Ae_k]_j = \sum_{j=1}^{n} a_{ij} e_{k,j} = a_{jk} \qquad \text{so} \qquad Ae_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix}$$

$$\|Ae_k\|_1 = \sum_{i=1}^{m} |a_{ik}| \qquad \|e_k\| = 0+0+\cdots 0+1+0+\cdots+0 = 1$$

$$\frac{\|Ae_k\|_1}{\|e_k\|_1} = \sum_{i=1}^{m} |a_{ik}| = M \qquad M = \sum_{i=1}^{m} |a_{ik}|$$

$$M = \frac{\|Ae_k\|_1}{\|e_k\|_1} \le \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \le M \qquad \text{(from before)}$$

$$\max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \|A\|_1$$

So $$\|A\|_1 = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

One more vector norm

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \lim_{p \to \infty} \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

$$= \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

There is a similar formula for the compatible matrix norm $\|A\|_\infty$ to what we just found.

For the compatible matrix norm $\|A\|_2$ we can find it with eigenvalues.

For values of $p \neq 1, 2$ or ∞, there are not exact formulas for finding the compatible matrix norm $\|A\|_p$. But algorithms for approximating exist...