$$\|A\|_p = \max\left\{ \frac{\|Ax\|_p}{\|x\|_p} \text{ for all } x \neq 0 \text{ and } x \in \mathbb{R}^n \right\}$$

what is this for $p=1$?

note rescaling x makes no difference in this quotient so just take it to be a unit vector.

$$\|A\|_p = \max\{ \|Ax\|_p : \|x\|_p = 1 \}$$

```
julia> using LinearAlgebra

julia> A=rand(4,4)
4×4 Matrix{Float64}:
 0.508788   0.465397  0.539458  0.300057
 0.0662677  0.279454  0.72069   0.559951
 0.63639    0.264154  0.490236  0.63369
 0.945861   0.894243  0.970979  0.244603
```

```
norm(A, p::Real=2)
```

```
\|A\|_p = \left( \sum_{i=1}^n | a_i | ^p \right)^{1/p}
```

$$\|A\|_p = \left(\sum_{i=1}^{m} |a_i|^p \right)^{1/p}$$

```
julia> x=rand(4)
4-element Vector{Float64}:
 0.5787635961096316
 0.06202225400378747
 0.9219974339296612
 0.25947347544357224

julia> u=x/norm(x,3)
4-element Vector{Float64}:
 0.5796642851376338
 0.062118774870076714
 0.923432273608882
 0.25987727574124225
```

unit vector

$$\|A\|_3 \geq \|Au\|_3$$

random unit vectors.

```
julia> norm(A*u,3)
1.8384893272996157
```

```
julia> norm(A*u,3)
1.9559102555399799
```

```
julia> for k=1:10
           x=rand(4)
           u=x/norm(x,3)
           println(norm(A*u,3))
       end
2.0580678281320957
2.224101543662481
2.164456508538842
2.202474121160809
2.11712388831675
1.6816974005295517
1.9410538463823854
1.9129890991390557
1.7401073445793804
1.8615036008381523
```

← largest (third largest automatically)

$\|A\|_3 \geq$ 2.224101543662481

↗ Gives a reasonable estimate of $\|A\|_3$, but not efficient...

For extra credit choose some values of $p = 3, 5, 13$ and estimate $\|A\|_p$ using the random approach shown above (maybe more than 10 trials).

Compare the estimate of $\|A\|_p$ to the estimate given by the complicated algorithm in...

N. Higham, Estimating the Matrix p-norm, Numerische Mathematik, Vol. 62, 1992, pp. 539-555.

(It's under class handouts)...

Idea is Solve $Ax = b$ for x. Approximation of x is x^*.

$$e = x^* - x \qquad \varepsilon = \|x^* - x\| \qquad \delta = \frac{\|x^* - x\|}{\|x\|}$$

I want to know $\frac{\|x^* - x\|}{\|x\|}$ ← don't know x so can't compute this

residual error... plug x^* into $Ax = b$ and see how wrong it is

let $b^* = Ax^*$ then $b^* \approx b$ residual error $\|b^* - b\|$

or relative residual error $\frac{\|b^* - b\|}{\|b\|}$

$$\frac{\|x^* - x\|}{\|x\|} = \frac{\|A^{-1}b^* - A^{-1}b\|}{\|x\|} = \frac{\|A^{-1}(b^* - b)\|}{\|x\|} \leq \frac{\|A^{-1}\|\,\|b^* - b\|}{\|x\|}$$

Compatible matrix norm

work on this

since $Ax = b$

then $\|b\| = \|Ax\| \leq \|A\|\,\|x\|$ so $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$

$$\frac{\|x^* - x\|}{\|x\|} \leq \|A^{-1}\|\,\|b^* - b\| \cdot \frac{\|A\|}{\|b\|} = \|A^{-1}\|\,\|A\|\,\frac{\|b^* - b\|}{\|b\|}$$

relating residual error

$$\frac{\|x^* - x\|}{\|x\|} \approx K_f(x)\,\frac{\|b^* - b\|}{\|b\|}$$

Condition number for the matrix is $K = \|A^{-1}\|\,\|A\|$

The reason this is useful is that it is easier to compute or estimate $\|A^{-1}\|$ than to find A^{-1} without the norm...

It good if condition number is small, If it's less than one that means error decrease, even.

$$\kappa = \|A^{-1}\|\|A\|$$

$$I = A^{-1}A$$

$$1 = \|I\| = \|A^{-1}A\| \leq \|A^{-1}\|\|A\| = \kappa$$

Thus $\kappa \geq 1$.

$$\frac{\|x^* - x\|}{\|x\|} \approx \kappa_f(x) \frac{\|b^* - b\|}{\|b\|}$$

This means if $\kappa_f(x) \approx 10^5$ we can't tell whether the last 5 digits of the approximation x^* is correct by plugging the answer back in.

Think about significant decimal digits.

Start with any floating point number $x^* \in F_{64}$ → round → to the nearest decimal with m significant digits → round → to the nearest floating point number $y^* \in F_{64}$

Question what is the smallest m such that $x^* = y^*$ for all $x^* \in F_{64}$ with the above path of roundings.

When Julia prints a floating point number, it prints enough digits so this round trip is an identity...

Start with any decimal number x with m significant digits → round → to the nearest floating point number → round → to the nearest decimal with m significant digits y.

Question what is the largest m such that $x = y$ for all x with m significant digits.