If A is invertible and $A = LDU$ where

L is lower triangular w/ ones on diagonal

D is diagonal

U is upper triangular w/ ones on diagonal

Then this decomposition is unique.. ☑

$L_2^{-1} L_1 = D U_2 U_1^{-1} D^{-1}$

lower triangular with ones on diagonal

upper triangular...

means $L_2^{-1} L_1$ is diagonal of ones

$L_2^{-1} L_1 = I$

so $L_2 = L_1$

$U_2 = U_1$

$D_2 = D_1$

What if A is also symmetric?

$A = A^T$ and invertible and $A = LDU$

$A = A^T = (LDU)^T = U^T D^T L^T = U^T D L^T$

$A = LDU$

U^T: lower triangular

L^T: upper triangular

Since LDU decomposition is unique the A being symmetric further implies that $L = U^T$ so $A = LDL^T$.

What if A is also positive definite.

Positive definite means $Ax \cdot x > 0$ for every $x \in \mathbb{R}^n$

$A = A^T$, A positive definite and $A = LDL^T$

• the existence of LDU factorization is automatic for positive definite matrix

$Ax \cdot x \geq 0$

$$LDL^T x \cdot x = \left(LDL^T x\right)^T x = \left((L)(DL^T x)\right)^T x$$

$$= (DL^T x)^T (L)^T x = DL^T x \cdot L^T x$$

easily invertible

let $y = L^T x$ so $x = (L^T)^{-1} y$

$$0 \leq Ax \cdot x = Dy \cdot y = \begin{bmatrix} d_1 y_1 \\ d_2 y_2 \\ \vdots \\ d_n y_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = d_1 y_1^2 + d_2 y_2^2 + \cdots + d_n y_n^2$$

Let $y = e_k$ then $y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ← k^{th} slot

$0 \leq De_k \cdot e_k = d_k$ so all the entries of D are non-negative... in fact they are positive since they can't be zero either...

$$D^{1/2} = \begin{bmatrix} d_1^{1/2} & & & 0 \\ & d_2^{1/2} & & \\ & & \ddots & \\ 0 & & & d_n^{1/2} \end{bmatrix}$$

$$D = D^{1/2} D^{1/2}$$

$$(D^{1/2})^T = D^{1/2}$$

$$A = LDL^T = LD^{1/2} D^{1/2} L^T = LD^{1/2} (D^{1/2})^T L^T = LD^{1/2} \left(LD^{1/2}\right)^T$$

let $\tilde{L} = LD^{1/2}$

Then $A = \tilde{L}\tilde{L}^T$ ← Cholesky decomposition...

If $A = A^T$ and A is positive definite, then there is a $\tilde{L}$ (lower triangular) so that $A = \tilde{L}\tilde{L}^T$

Note that usually the definition of pos. definite requires that A be symmetric...

```
for i=1:n
    s=0
    for k=1:i-1
        s+=L[i,k]^2
    end
    if A[i,i]-s <= 0
        error("input is not positi
    end
    L[i,i]=sqrt(A[i,i]-s)
    for j=i+1:n
        s=0
        for k=1:i-1
            s+=L[j,k]*L[i,k]
        end
        L[j,i]=(A[j,i]-s)/L[i,i]
    end
end
```

$$s = \sum_{k=1}^{i-1} L_{ik}^2$$

```
s=sum(L[i,k]^2 for k=1:i-1; init=0)
```

empty sum

$$s = \sum_{k=1}^{i-1} L_{jk} L_{ik}$$

```
s=sum(L[j,k]*L[i,k] for k=1:i-1; init=0)
```

empty sum

Trying to find Cholesky decomposition is a good way to check if a matrix is positive definite..

$Ax \cdot x \geq 0$ for all $x \in \mathbb{R}^n$

About the Matrix 2-norm...

$$\|A\|_2 = \max\{\|Ax\|_2 : \|x\|_2 = 1\}$$

matrix norm. vector norms

$$\|Ax\|_2^2 = Ax \cdot Ax = A^TAx \cdot x = Bx \cdot x$$

Let $B = A^TA$ symmetric and positive definite...

Spectral Theorem for Symmetric Matrices: Let $B \in \mathbb{R}^{n\times n}$.

Suppose $B = B^T$. Then B has an orthonormal basis of eigenvectors. Thus

$$Bu_k = \lambda_k u_k \quad \text{for } k = 1, 2, \dots, n$$

and $$u_k \cdot u_\ell = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

$$x = \sum_{i=1}^{n} c_i u_i \qquad Bx = \sum_{i=1}^{n} c_i B u_i = \sum_{i=1}^{n} \lambda_i c_i u_i$$

$$Bx \cdot x = \sum_{i=1}^{n} \lambda_i c_i u_i \cdot \sum_{j=1}^{n} c_j u_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i c_i c_j u_i \cdot u_j$$

orthonormal

only terms that survive is $i = j$

$$\| Ax \|_2^2 = \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\| x \|^2 = \sum_{i=1}^{n} c_i u_i \cdot \sum_{j=1}^{n} c_j u_j = \sum_{i=1}^{n} c_i^2 = 1$$

since $\sum_{i=1}^{n} c_i^2 = 1$ then $\sum_{i=1}^{n} \lambda_i c_i^2$ can be interpreted as a weighted average of the λ_i's.

$Bu_k = \lambda_k u_k$

orthonormal

$$0 \leq \| Au_k \|_2^2 = Bu_k \cdot u_k = \lambda_k u_k \cdot u_k = \lambda_k \| u_k \|_2^2 = \lambda_k$$

$$\| A \|_2^2 = \max\{ \| Ax \|_2^2 : \| x \|_2^2 = 1 \} = \max\{ \lambda_k : \lambda_k \text{ is an eigenvalue of } A^T A \}$$

B

$$\|A\|_2 = \max\left\{ \lambda_k^{1/2} : \lambda_k \text{ is an eigenvalue of } A^T A \right\}$$

$$\|A\|_2 = \max\left\{ \sigma_k : \sigma_k \text{ is a singular value of } A \right\}$$

(def. of $\sigma_k = \sqrt{\lambda_k}$)

QR decomposition by Givens rotations next time...