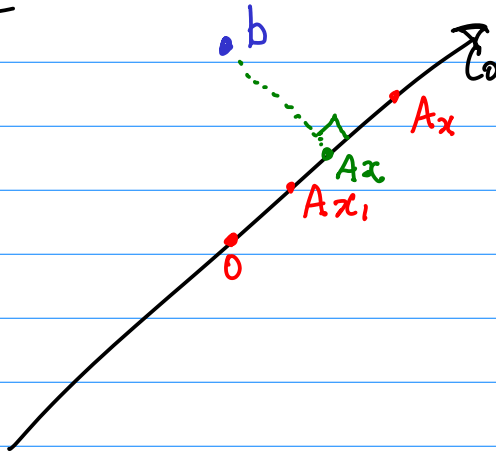
How to "solve" $Ax = b$ when there is no solution?

Minimize the norm $\|Ax - b\|_2$:

since b is not in Col A the equation $Ax = b$ can't be solved.



since Q is orthogonal

$A = QR$ $\quad Q^T = Q^{-1}$

$QRx = b$

$Rx = Q^T b$

```
julia> include("givens.jl")
givens (generic function with 1 method)

julia> A=rand(7,3)
7×3 Matrix{Float64}:
 0.0283123  0.977688   0.717241
 0.0182588  0.698964   0.776749
 0.699655   0.343708   0.931254
 0.369735   0.0317075  0.142078
 0.728      0.642568   0.20725
 0.814941   0.493196   0.810486
 0.994541   0.557831   0.558543

julia> A=A.-0.5
7×3 Matrix{Float64}:
 -0.471688   0.477688    0.217241
 -0.481741   0.198964    0.276749
  0.199655  -0.156292    0.431254
 -0.130265  -0.468293   -0.357922
  0.228      0.142568   -0.29275
  0.314941  -0.00680361  0.310486
  0.494541   0.0578308   0.0585428
```

```
julia> b=rand(7).-0.5
7-element Vector{Float64}:
  0.227395160099849 84
  0.282955305425517 17
  0.103268803764057 78
  0.272276792342348 25
 -0.127309777741985 15
  0.266222361018823 35
  0.348531526152191 85
```

$A \in \mathbb{R}^{7\times 3}$ $\quad b \in \mathbb{R}^7$

want to find $x \in \mathbb{R}^3$ such that $\|Ax - b\|_2$ is minimal...

```
julia> RA,Rb=givens(A,b);

julia> RA
7×3 Matrix{Float64}:
 0.952438      -0.244014      -0.045226
-1.52666e-18    0.689689       0.300929
 3.40836e-17    4.6199e-18     0.72856
 4.22461e-18    8.01106e-19    1.38778e-17
-1.26549e-17    3.29521e-18    0.0
-1.31752e-17    1.26392e-17    2.77556e-17
 4.83349e-17    0.0            0.0

julia> Rb
7-element Vector{Float64}:
 -0.03280011614784356
  0.019526544610845905
  0.28516667973065685
  0.5218791430305046
 -0.21609256543100852
 -0.009291881116494927
  0.15234948739545512
```

$$R = \begin{bmatrix} U_{3\times 3} \\ \hline 0_{4\times 3} \end{bmatrix}$$

$$Q^T b = \begin{bmatrix} y \\ \hline ? \end{bmatrix}$$

```
julia> RA[1:3,1:3]
3×3 Matrix{Float64}:
  0.952438      -0.244014      -0.045226
 -1.52666e-18    0.689689       0.300929
  3.40836e-17    4.6199e-18     0.72856
```

← submatrix I called U.

```
julia> Rb[1:3]
3-element Vector{Float64}:
 -0.03280011614784356
  0.019526544610845905
  0.28516667973065685
```

```
julia> include("backward.jl")
backward (generic function with 1 method)
```

```
julia> backward(RA[1:3,1:3],Rb[1:3])
3-element Vector{Float64}:
 -0.05235317007688747
 -0.14247100452553269
  0.3914115227994955
```

This value minimizes $\|Ax-b\|$

```
julia> norm(A*x1-b)
0.585107291126294
```

```
julia> norm(A*x1-b)
0.585107291126294

julia> norm(Rb[4:7])
0.5851072911262939

julia> x2=A\b
3-element Vector{Float64}:
 -0.05235317007688743
 -0.14247100452553252
  0.391411522799495 26
```

The part of $Q^T b$ that we threw away is the distance from b to Ax.

The same left division by A automatically switches to least squares solution when $Ax=b$ has no exact solution...

Goal: Solving $Ax=b$ when there is a solution...

2.11 Iterative Schemes and Splitting

$A \in \mathbb{R}^{n \times n}$

Idea: Write $A = M - N$ ← the rest of A.

M is easy to invert

$$(M-N)x = b$$

So $$Mx = Nx + b$$

Iterate: Set $x^{(0)} = 0 \in \mathbb{R}^n$

$$M x^{(k+1)} = N x^{(k)} + b$$

hope this converges to the solution.

Jacobi Iterations.

leftovers stuff

$M = \text{Diagonal}(A)$: $N = M - A$

```
julia> A=rand(5,5).-0.5
5×5 Matrix{Float64}:
 -0.0539841   0.302683  -0.232992     0.221852    0.000308454
  0.285801   -0.346909  -0.216222    -0.0884532  -0.355179
  0.334609   -0.269749  -0.190517     0.0463425   0.0861036
 -0.141337    0.134789  -0.11251     -0.0465706   0.0324983
  0.218138   -0.121991   0.00335043  -0.172861    0.214201
```

```
julia> M=Diagonal(A)
5×5 Diagonal{Float64, Vector{Float64}}:
 -0.0539841    ⋅           ⋅           ⋅           ⋅
   ⋅          -0.346909    ⋅           ⋅           ⋅
   ⋅            ⋅         -0.190517    ⋅           ⋅
   ⋅            ⋅           ⋅         -0.0465706   ⋅
   ⋅            ⋅           ⋅           ⋅          0.214201
```

Gauss-Seidel Iterations

$M = \text{lowerTriangular}(A)$: $N = M - A$

```
julia> M=LowerTriangular(A)
5×5 LowerTriangular{Float64, Matrix{Float64}}:
 -0.0539841    ⋅           ⋅           ⋅           ⋅
  0.285801   -0.346909     ⋅           ⋅           ⋅
  0.334609   -0.269749  -0.190517      ⋅           ⋅
 -0.141337    0.134789  -0.11251     -0.0465706   ⋅
  0.218138   -0.121991   0.00335043  -0.172861    0.214201
```

Tridiagonal Jacobi Iterations

M = Tridiagonal(A) ; N = M - A

```
julia> M=Tridiagonal(A)
5×5 Tridiagonal{Float64, Vector{Float64}}:
 -0.0539841   0.302683    .          .          .
  0.285801   -0.346909   -0.216222   .          .
   .         -0.269749   -0.190517   0.0463425  .
   .           .         -0.11251   -0.0465706  0.0324983
   .           .           .        -0.172861   0.214201
```

When does it converge?

Definition 2.6 (Spectral radius). *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix H. Then its spectral radius $\rho(H)$ is defined as*

$$\rho(H) = \max_{i=1,\ldots,n} (|\lambda_i|).$$

Definition 2.7 (Strictly diagonally dominant matrices). *A matrix A is called strictly diagonally dominant by rows if*

$$|A_{i,i}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{i,j}|$$

for $i, 1, \ldots, n$.

Theorem: If A is strictly diagonally dominant, then all the 3 splitting methods for solving $Ax = b$ converge to the solution.

Definition 2.7 (~~Strictly~~ Weakly diagonally dominant matrices). *A matrix A is called ~~strictly~~ weakly diagonally dominant by rows if*

$$|A_{i,i}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{i,j}|$$

for $i, 1, \ldots, n$.

Theorem: If A is weakly diagonally dominant, then all the 3 splitting methods for solving $Ax = b$ converge to the solution.

provided for at least one row of each irreducible part of the matrix A there is a strict inequality..

$$A = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ 0 & & \ddots & \ddots \\ & & & 1 & -2 \end{bmatrix} \leftarrow$$

$|2| > |1|$ ← actually 2 rows

$|2| = |1| + |1|$

$\vdots$

$|2| > |1|$

Let's look more closely:

$$M x^{(k+1)} = N x^{(k)} + b$$

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b$$

$$d_k = x^{(k+1)} - x^{(k)}$$

$$d_{k+1} = x^{(k+2)} - x^{(k+1)}$$

$$= M^{-1}N x^{(k+1)} + M^{-1}b - \left(M^{-1}N x^{(k)} + M^{-1}b\right)$$

$$= M^{-1}N\left(x^{(k+1)} - x^{(k)}\right) = M^{-1}N\, d_k .$$

For example, lets look at norms.

$$\|d_{k+1}\| = \|M^{-1}N\, d_k\| \leq \|M^{-1}N\|\,\|d_k\|$$

If $\|M^{-1}N\| < 1$ then the iterations converge...

Definition 2.6 (Spectral radius). *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix H. Then its spectral radius $\rho(H)$ is defined as*

$$\rho(H) = \max_{i=1,\ldots,n} (|\lambda_i|).$$

The iterations converge if and only if $\rho(M^{-1}N) < 1$.

Necessary since if there is an eigendirection (vector) which is expanding, there is trouble.