- Recall what A^T means the matrix such that

$$Ax \cdot y = x \cdot A^T y \quad \text{for all } x, y \in \mathbb{R}^n$$

- Symmetric Matrix

$$Ax \cdot y = x \cdot Ay \quad \text{for all } x, y \in \mathbb{R}^n$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Standard basis vectors for $\mathbb{R}^n$.

e_k is the vector which is 1 in the k^{th} element and 0 everywhere else..

$$Ae_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \simeq \begin{bmatrix} a_{1k} \\ a_{2k} \\ \\ \\ a_{nk} \end{bmatrix}$$

k^{th} element

$$Ae_k \cdot e_l = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \\ a_{nk} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = a_{lk}$$

l^{th} element

Positive definite: $A \in \mathbb{R}^{n \times n}$ (and $A^T = A$)

$Ax \cdot x > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$

What about complex numbers and vectors?

Suppose $A \in \mathbb{R}^{n \times n}$ and $A^T = A$ and A is positive definite

Let $z \in \mathbb{C}^n$. Then $z = x + iy$ where $x, y \in \mathbb{R}^n$.

I know that $Ax \cdot x > 0$ and $Ay \cdot y > 0$

if $x \neq 0$ if $y \neq 0$

If $z \neq 0$ then at least one of x or y is not 0.

Therefore $Ax \cdot x + Ay \cdot y > 0$

$$Ax \cdot x - i^2 Ay \cdot y > 0$$

$$Ax \cdot x - Aiy \cdot iy > 0$$

$$A(x+iy) \cdot (x-iy) = Ax \cdot x - Ax \cdot iy + Aiy \cdot x - Aiy \cdot iy$$

since A is symmetric.

$$Ax \cdot iy = x \cdot A^T iy = A^T iy \cdot x = Aiy \cdot x$$

Therefore $Az \cdot \bar{z} > 0$ for all $z \in \mathbb{C}^n$.

Exercise 2.12. *Prove that the Gauss–Seidel method to solve* $A\mathbf{x} = \mathbf{b}$ *converges whenever the matrix* A *is symmetric and positive definite. Show how-*

Gauss-Seidel method:

$A = M - N$ ← what's left

↑ easy to invert

$M = \text{LowerTriangular}(A)$; $N = M - A$

iterative scheme to solve $\boxed{Ax = b}$ ← assume A is invertible

$$M x^{(k+1)} = N x^{(k)} + b$$

Explicitly

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b$$

We are interested in $\rho(M^{-1}N)$ so what are eigenvalues and eigenvectors? Let λ and v be such that

eigenvector ↓ eigenvalue ↓ note $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$

$$M^{-1}N v = \lambda v \qquad \text{so} \qquad Nv = \lambda M v$$

$$Nv = \lambda M v$$
$$-\lambda N v = -\lambda N v$$
$$(1-\lambda) N v = \lambda (M-N) v = \lambda A v$$

since A is invertible and $v \neq 0$ because it's an eigenvector then $Av \neq 0$.

the only way this could be 0 is if $\lambda = 0$.

Claim $\lambda \neq 1$. For contradiction, suppose $\lambda = 1$.

then $\lambda A v \neq 0$ but $\lambda A v = (1-\lambda) N v = 0$

but this is 0 if $\lambda = 1$

This is a contradiction. Thus $\lambda \neq 1$.

$$Nv = \frac{\lambda}{1-\lambda} Av$$

$$N\bar{v} = \frac{\bar{\lambda}}{1-\bar{\lambda}} A\bar{v}$$

$$\overline{Nv} = \frac{\overline{\lambda}}{1-\overline{\lambda}} Av$$

$$\overline{N}\bar{v} = \frac{\bar{\lambda}}{1-\bar{\lambda}} \bar{A}\bar{v}$$

Hypothesis of theorem

Let's assume $A = A^T$ and positive definite.

$$Nv \cdot \bar{v} = \frac{\lambda}{1-\lambda} Av \cdot \bar{v}$$

$$N\bar{v} \cdot v = \frac{\bar{\lambda}}{1-\bar{\lambda}} A\bar{v} \cdot v = \frac{\bar{\lambda}}{1-\bar{\lambda}} \bar{v} \cdot Av = \frac{\bar{\lambda}}{1-\bar{\lambda}} Av \cdot \bar{v}$$

$$\| \quad \bar{v} \cdot N^T v = N^T v \cdot \bar{v}$$

Therefore

$$Nv \cdot \bar{v} = \frac{\lambda}{1-\lambda} Av \cdot \bar{v}$$

and

$$N^T v \cdot \bar{v} = \frac{\bar{\lambda}}{1-\bar{\lambda}} Av \cdot \bar{v}$$

$$(N + N^T) v \cdot \bar{v} = \left(\frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}} \right) Av \cdot \bar{v}$$

$$(N + N^T) v \cdot \bar{v} = \frac{\lambda(1-\bar{\lambda}) + \bar{\lambda}(1-\lambda)}{(1-\lambda)(1-\bar{\lambda})} Av \cdot \bar{v}$$

$$(1-\lambda)(1-\bar{\lambda}) = |1-\lambda|^2$$

$$\lambda(1-\bar{\lambda}) + \bar{\lambda}(1-\lambda) = \lambda - \lambda\bar{\lambda} + \bar{\lambda} - \lambda\bar{\lambda} = \lambda + \bar{\lambda} - 2|\lambda|^2$$

$$Av\cdot\bar{v} + (N+N^T)v\cdot\bar{v} = \frac{\lambda(1-\bar{\lambda}) + \bar{\lambda}(1-\lambda)}{(1-\lambda)(1-\bar{\lambda})}\, Av\cdot\bar{v} + Av\cdot\bar{v}$$

$$(A+N+N^T)v\cdot\bar{v} = \left(1 + \frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}}\right) Av\cdot\bar{v}$$

$$1 + \frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}} = \frac{1-\lambda}{1-\lambda} + \frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}} = \frac{1}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}}$$

$$= \frac{1-\bar{\lambda} + \bar{\lambda}(1-\lambda)}{|1-\lambda|^2} =$$

Next step would be showing this is positive.

$$(A+N+N^T)v\cdot\bar{v} = \frac{1-|\lambda|^2}{|1-\lambda|^2}\, Av\cdot\bar{v}$$

if I know this is positive

know this is positive already since A is positive definite

has to be positive

$1 - |\lambda|^2 > 0$

so $|\lambda|^2 < 1$ or $|\lambda| < 1$

Thus $\rho(M^{-1}N) < 1$.

Therefore, except for showing that $(A+N+N^T)v\cdot\bar{v} > 0$ we've shown that Gauss-Seidel converges.