Gauss-Seidel method:

$A = M - N$ ← what's left

↑ easy to invert

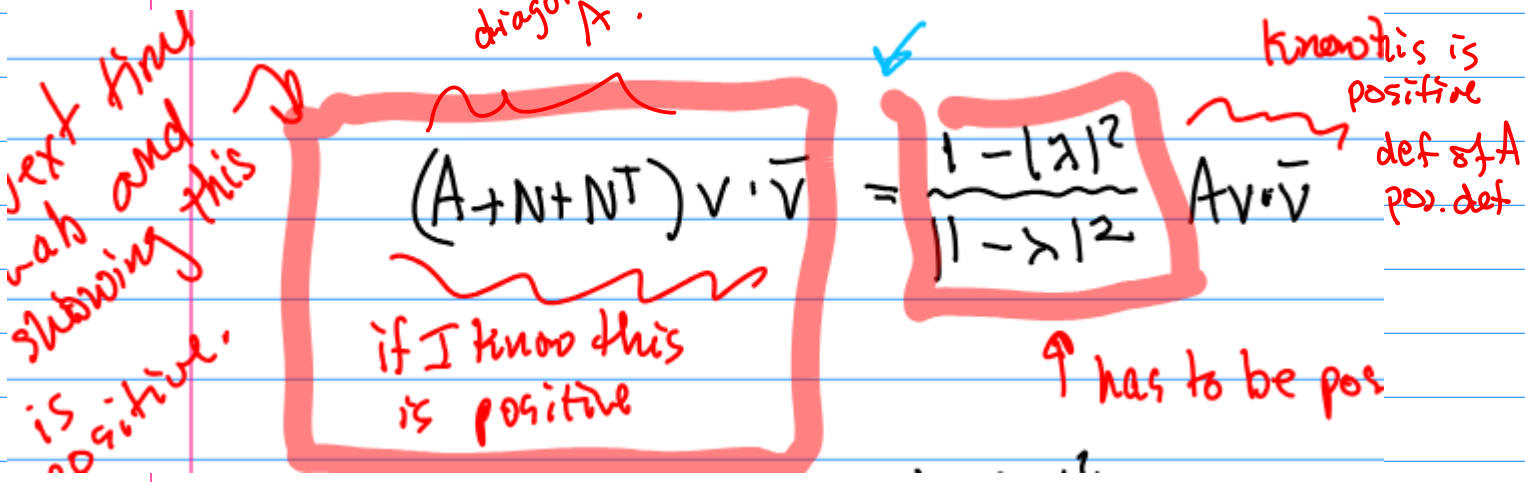
$M = \text{lowerTriangular}(A)$; $N = M - A$

iterative scheme to solve $\boxed{Ax = b}$ ← assume A is invertible

$$M x^{(k+1)} = N x^{(k)} + b$$

The eigenvalues of the matrix $M^{-1}N$ satisfy

Show: If A is positive definite and symmetric then $\rho(M^{-1}N) < 1$



$$A = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \quad M = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix} \quad N = M - A = \begin{bmatrix} 0 & -x & -x \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{bmatrix}$$

$A = M + N + N^T$

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} + \begin{bmatrix} 0 & -x & -x \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -x & 0 & 0 \\ -x & -x & 0 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

Diagonal of a Positive definite matrix is positive

$Ax \cdot x > 0$ for all $x \in \mathbb{R}^n$

$Ae_i \cdot e_i = a_{ii} > 0$

↖ diagonal.

Now since $A+N+N^T$ is a diagonal matrix with all positive entries, then

from last time ...

$$Ae_k \cdot e_l = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \\ a_{nk} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \leftarrow l^{th} \text{ element} = a_{lk}$$

$$(A+N+N^T)v \cdot \bar{v} = \sum_{i=1}^{n} a_{ii} v_i \bar{v}_i = \sum_{i=1}^{n} a_{ii} |v_i|^2 > 0$$