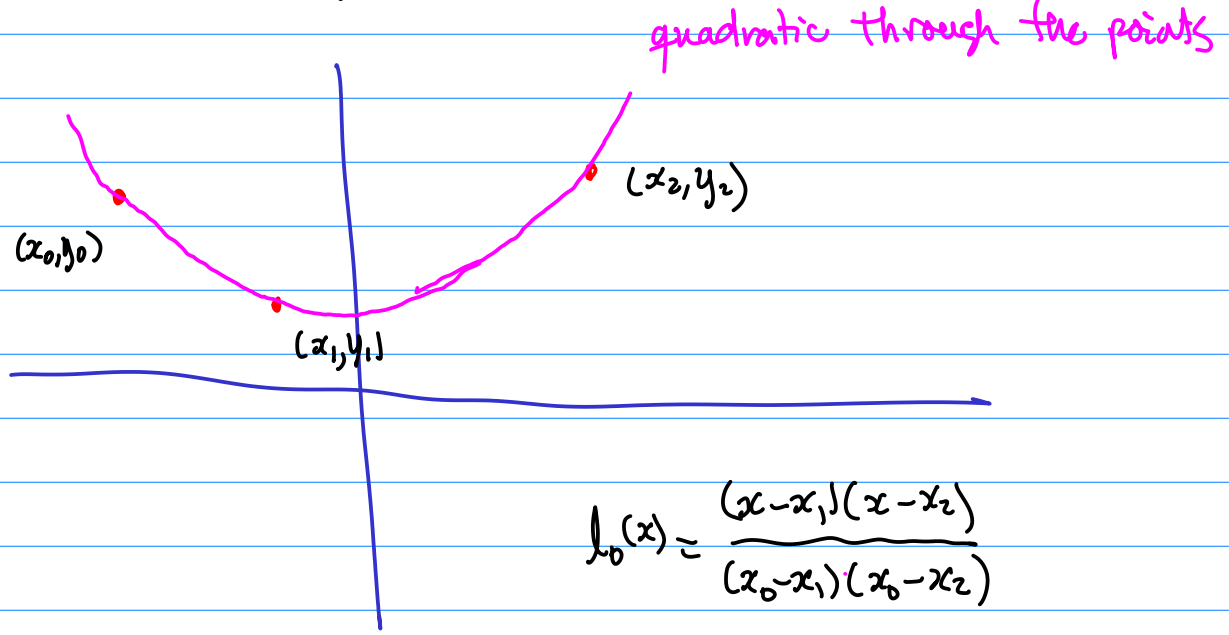
Want to interpolate these points

(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)

with an n-degree polynomial.



$$\ell_k(t) = \prod_{i \neq k} \frac{x_i - t}{x_i - x_k}$$

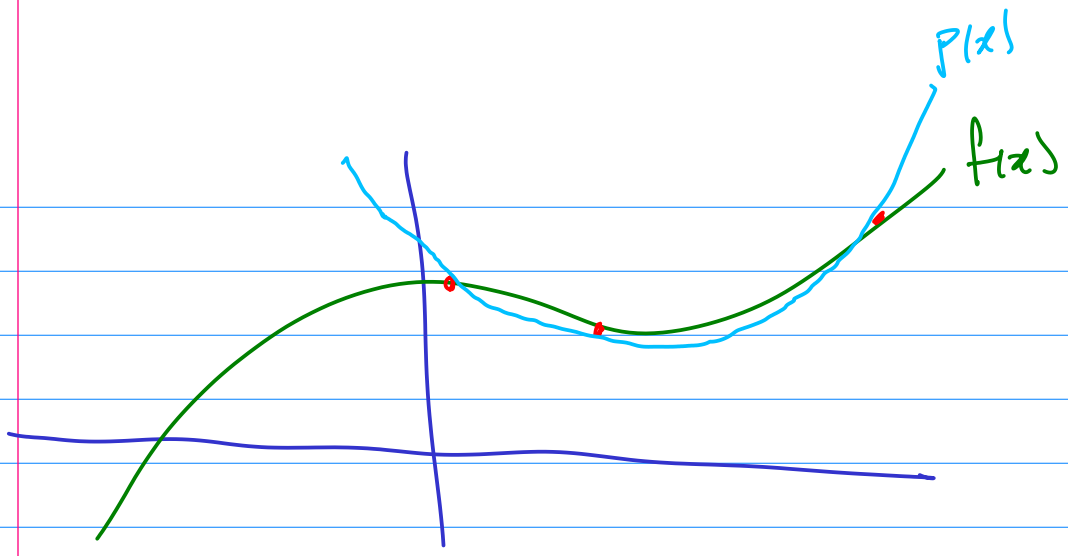
$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_0)}$$

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$$

Suppose $y_k = f(x_k)$ then maybe the $p(x)$ approximates the function $f(x)$.



Use Calculus to estimate the error $E(x) = p(x) - f(x)$.

$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$

degree n

$$p(x) = \sum_{k=0}^{n} f(x_k)\, \ell_k(x)$$

polynomials of degree n.

$$\ell_k(t) = \prod_{i \neq k} \frac{x_i - t}{x_i - x_k}$$

How good is the approximation $p(x) \approx f(x)$?

$$q(x) = \prod_{k=0}^{n} (x - x_k)$$

polynomial of degree $n+1$

Note that $q(x_k) = 0$ for $k = 0, \ldots, n$
and also $E(x_k) = 0$ for $k = 0, \ldots, n$
So maybe there is a relation?

$$\frac{d^{n+1}}{dx^{n+1}} q(x) = \frac{d^{n+1}}{dx^{n+1}} x^{n+1} = (n+1)\frac{d^n}{dx^n} x^n = \ldots = (n+1)!$$

$$F(x) = E(x) - \alpha\, q(x)$$

Let's suppose we want to estimate $E(t)$ for some t fixed.
If $t \neq x_k$ for $k = 0, \ldots, n$ then $q(t) \neq 0$.

Use the point t and $g(t)$ to determine the scale α.
If I've chosen the scale correctly, then...

$$E(t) - \alpha g(t) = 0$$

Note α depends on t, but we've fixed t (for now).

$$\alpha = \frac{E(t)}{g(t)}$$

Therefore

$$F(x) = E(x) - \frac{E(t)}{g(t)} g(x)$$

Now let's think about F

$n+1$ zeros

$$F(x_k) = E(x_k) - \frac{E(t)}{g(t)} g(x_k) = 0$$

zero ; zero

1 zero

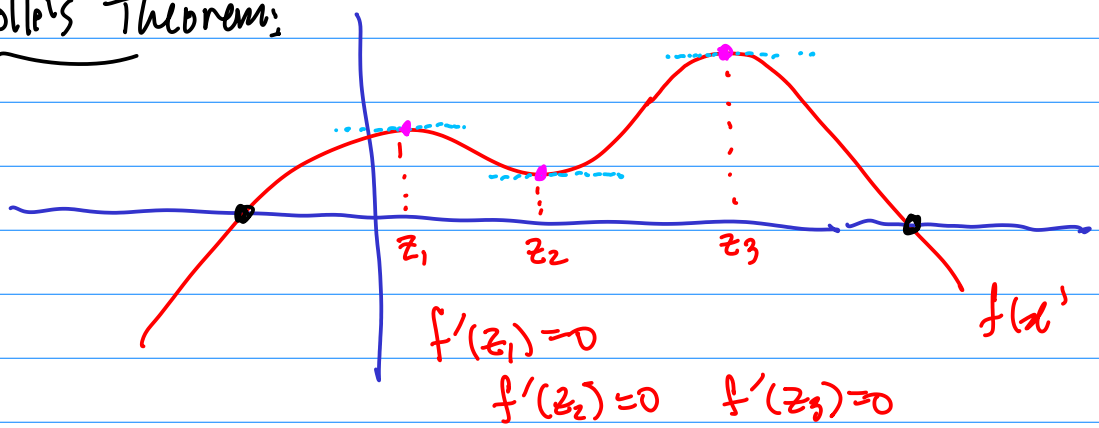
$$F(t) = E(t) - \frac{E(t)}{g(t)} g(t) = 0$$

means that F has $n+2$ places where it's zero.

Where are the zeros? They are in the interval $[a,b]$ where

$$a = \min(x_0, x_1, \ldots, x_n, t) \qquad b = \max(x_0, x_1, \ldots, x_n, t).$$

Rolle's Theorem:



Between any two zeros of a differentiable function the derivative has at least one zero.

$F(x)$ has $n+2$ zeros on $[a,b]$

$F'(x)$ has at least $n+1$ zeros on $[a,b]$

$F''(x)$ has at least n zeros on $[a,b]$

$\vdots$

$F^{(n+1)}(x)$ has at least 1 zero on $[a,b]$.

Let $c \in [a,b]$ be such that $F^{(n+1)}(c) = 0$.

$$F(x) = E(x) - \frac{E(t)}{q(t)} q(x)$$

recall

$$\frac{d^{n+1}}{dx^{n+1}} q(x) = (n+1)!$$

$$F^{(n+1)}(x) = E^{(n+1)}(x) - \frac{E(t)}{q(t)} q^{(n+1)}(x)$$

$q^{(n+1)}(x) = (n+1)!$

$$E^{(n+1)}(x) = p^{(n+1)}(x) - f^{(n+1)}(x).$$

$p^{(n+1)}(x)$: zero since p is a poly of degree n.

$$F^{(n+1)}(x) = -f^{(n+1)}(x) - \frac{E(t)}{q(t)} (n+1)!$$

$$0 = F^{(n+1)}(c) = -f^{(n+1)}(c) - \frac{E(t)}{q(t)} (n+1)!$$

Now solve for $E(t)$, because that's what we are trying to find...

$$-f^{(n+1)}(c) - \frac{E(t)}{q(t)}(n+1)! = 0$$

$$\frac{E(t)}{q(t)}(n+1)! = -f^{(n+1)}(c)$$

$$E(t) = \frac{-q(t)}{(n+1)!} f^{(n+1)}(c)$$

I found the error

Since:

$$E(t) = p(t) - f(t) \qquad q(t) = \prod_{k=0}^{n} (t - x_k)$$

then

$$p(t) - f(t) = \frac{\prod_{k=0}^{n} (t - x_k)}{(n+1)!} f^{(n+1)}(c)$$

Therefore

$$f(t) = p(t) + \frac{\prod_{k=0}^{n} (t - x_k)}{(n+1)!} f^{(n+1)}(c)$$

Polynomial interpolation theorem: Let f have $n+1$ derivatives and $x_0, x_1, \ldots, x_n$ have distinct values. Let $p(t)$ be the interpolating polynomial through the points

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots (x_n, f(x_n))$$

Thm

$$f(t) = p(t) + \frac{\prod_{k=0}^{n} (t - x_k)}{(n+1)!} f^{(n+1)}(c)$$

← polynomial in t of degree $n+1$

for some c between $\min(t, x_0, \ldots, x_n)$ and $\max(t, x_0, \ldots, x_n)$.

Taylor's Theorem: Let f have $n+1$ derivatives and $p(t)$ be the Taylor polynomial at x_0.

$$p(t) = \sum_{k=0}^{n} \frac{(t - x_0)^k}{k!} f^{(k)}(x_0)$$

← Taylor polynomial of degree n.

Then

$$f(t) = p(t) + \frac{(t - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

for some c between $\min(t, x_0)$ and $\max(t, x_0)$.

Similar techniques prove all sorts of polynomial approximation theorems. In general we have peano kernel theorem.