**Theorem 3.1** (The error of polynomial interpolation). *Given $f \in C^{n+1}[a,b]$ and $f(x_i) = f_i$, where $x_0, \ldots, x_n$ are pairwise distinct, let $p \in \mathbb{P}_n[x]$ be the interpolating polynomial. Then for every $x \in [a,b]$, there exists $\xi \in [a,b]$ such that*

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \underbrace{\prod_{i=0}^{n} (x - x_i)}_{q(x)}. \tag{3.1}$$

From last time

Idea change the interpolation points

$(x_0, f(x_0)), \cdots, (x_n, f(x_n))$

to make $q(x)$ smaller on the domain $[-1,1]$.

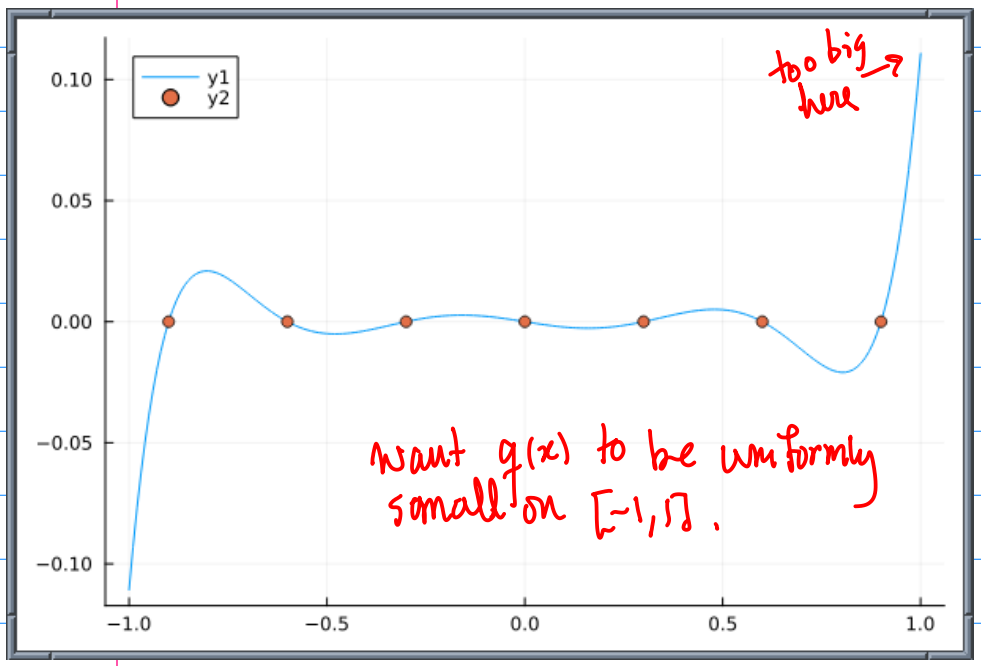
```
julia> using Plots

julia> q(t)=prod(t.-x)
q (generic function with
```

```
julia> x=[-.9:.3:.9;]
7-element Vector{Float64}:
 -0.9
 -0.6
 -0.3
  0.0
  0.3
  0.6
  0.9
```

```
julia> plot(q,-1:0.01:1)

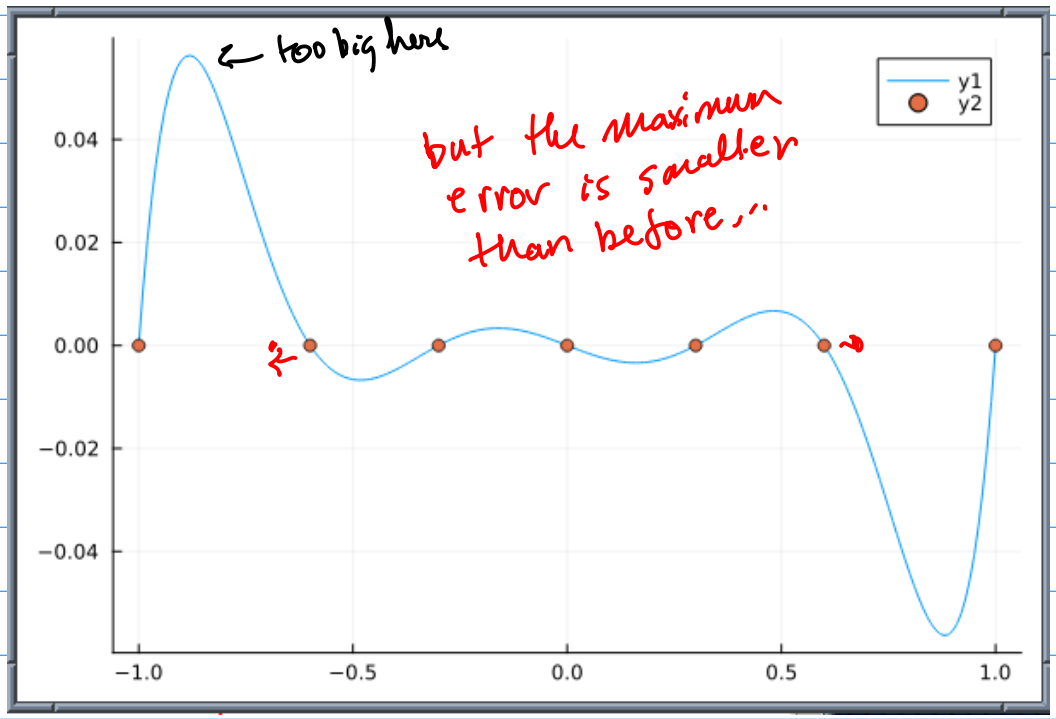
julia> scatter!(x,zeros(7))
```



```
julia> x[1]=-1; x[7]=1; x
7-element Vector{Float64}:
 -1.0
 -0.6
 -0.3
  0.0
  0.3
  0.6
  1.0
```

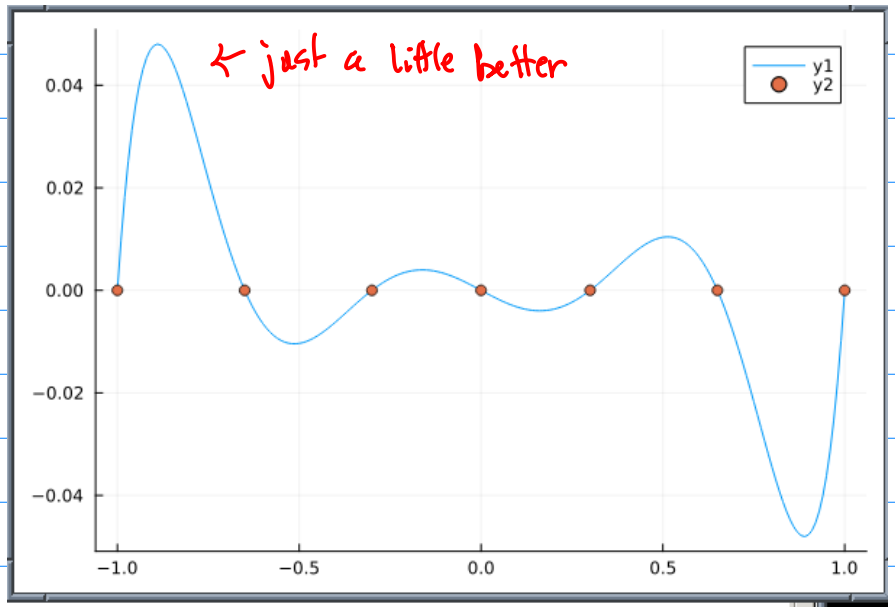
```
julia> plot(q,-1:0.01:1)

julia> scatter!(x,zeros(7))
```




Choose the points $x_i$'s so that $\max\{|p(x)| : x \in [-1,1]\}$ is as small as possible.

```
julia> x[2]=-0.65; x[6]=0.65; x
7-element Vector{Float64}:
 -1.0
 -0.65
 -0.3
  0.0
  0.3
  0.65
  1.0
```



The answer is

These Chebyshev points are the zeros of the $(n+1)^{\text{th}}$ *Chebyshev polynomial*, which is defined on the interval $[-1, 1]$ by the *trigonometric definition*

$$T_{n+1}(x) = \cos((n+1)\arccos x), \qquad n \geq -1.$$

↑ polynomial of degree $n+1$

$T_0(x) = \cos(0 \cdot \arccos x) = \cos 0 = 1$

$T_1(x) = \cos(1 \cdot \arccos x) = x$

$T_2(x) = \cos(2\arccos x) = \cos(2\theta) = 2\cos^2\theta - 1 = 2x^2 - 1$

$\theta = \arccos x$

$\sin(a+b) = \sin a\cos b + \cos a \sin b$

$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a\cos b + \cos a \sin b)$

$\cos^2\theta + \sin^2\theta = 1$

$-\sin^2\theta = \cos^2\theta - 1$

$\cos(a+b) = \cos a\cos b - \sin a \sin b$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\cos(2\theta) = 2\cos^2\theta - 1$

$T_{n+1}(x) = \cos((n+1)\arccos x), = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$ ($\theta$)

$= T_n(x)\,x - \sin(n\theta)\sin\theta$ odd functions

$T_{n-1}(x) = \cos((n-1)\theta) = \cos(n\theta)\cos(-\theta) - \sin(n\theta)\sin(-\theta)$

$= T_n(x)\,x + \sin(n\theta)\sin\theta$

---

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

Therefore

the coef on the leading term is a power of 2.

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

recall

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$

$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2-1) - x$

$= 4x^3 - 2x - x = 4x^3 - 3x$

$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1)$

$2^3$ $= 8x^4 - 8x^2 + 1$

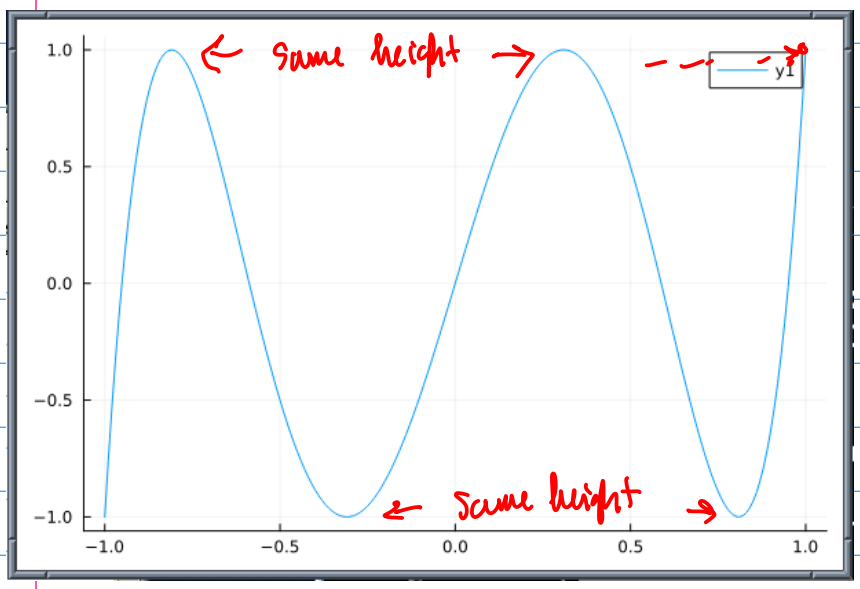
$T_5(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$

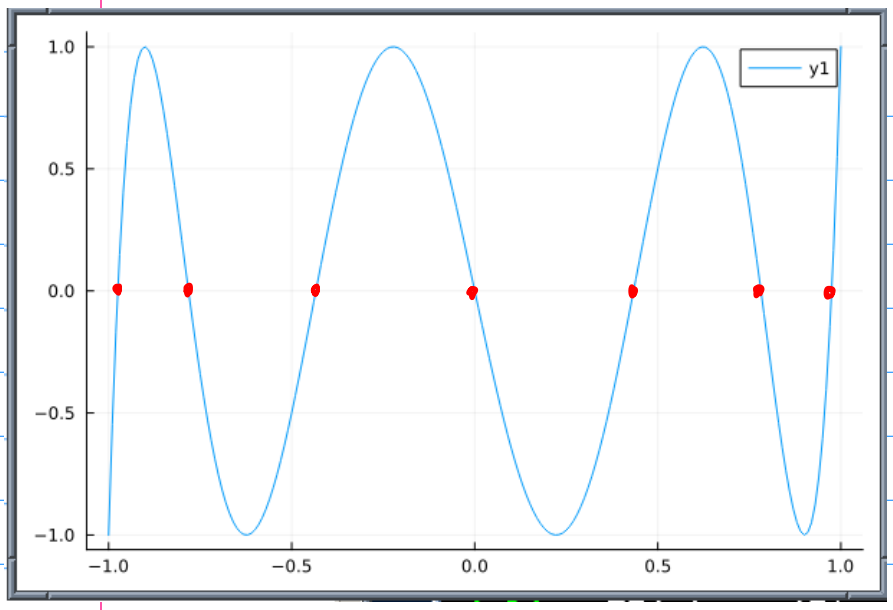
$2^4$ $= 16x^5 - 20x^3 + 5x = (16x^4 - 20x^2 + 5)x$

$= ((16x^2 - 20)x^2 + 5)x$

```
julia> T5(x)=((16*x^2-20)*x^2+5)*x
T5 (generic function with 1 method)
```

```
julia> plot(T5,-1:0.01:1)
```






```
julia> T7(x)=cos(7*acos(x))
T7 (generic function with 1 method)

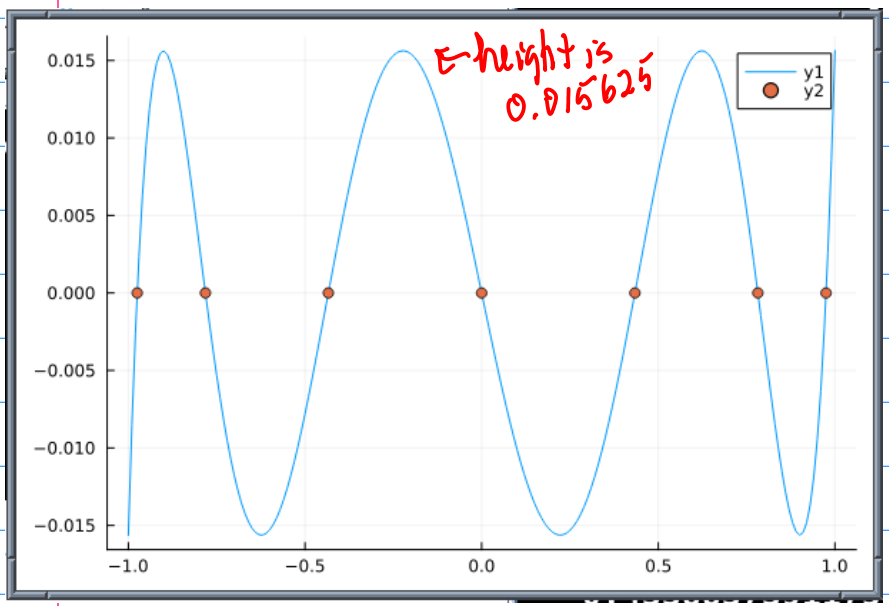
julia> plot(T7,-1:0.01:1)
```

$$x_k = \cos\left(\frac{\pi}{2}\frac{2k+1}{n+1}\right),$$

$$x_k = \cos\left(\frac{\pi}{2}\frac{2k+1}{7}\right)$$

$T_{n+1}(x)$ is a polynomial with $2^n x^{n+1}$ as the leading term

```
julia> x=[cos(pi/2*(2*k+1)/7) for k=0:6]
7-element Vector{Float64}:
  0.9749279121818236
  0.7818314824680298
  0.4338837391175582
  6.123233995736766e-17
 -0.43388373911755806
 -0.7818314824680295
 -0.9749279121818237
```




```
julia> 1/2^6
0.015625
```

4 times smaller than the $q(x)$ that was our best attempt before..

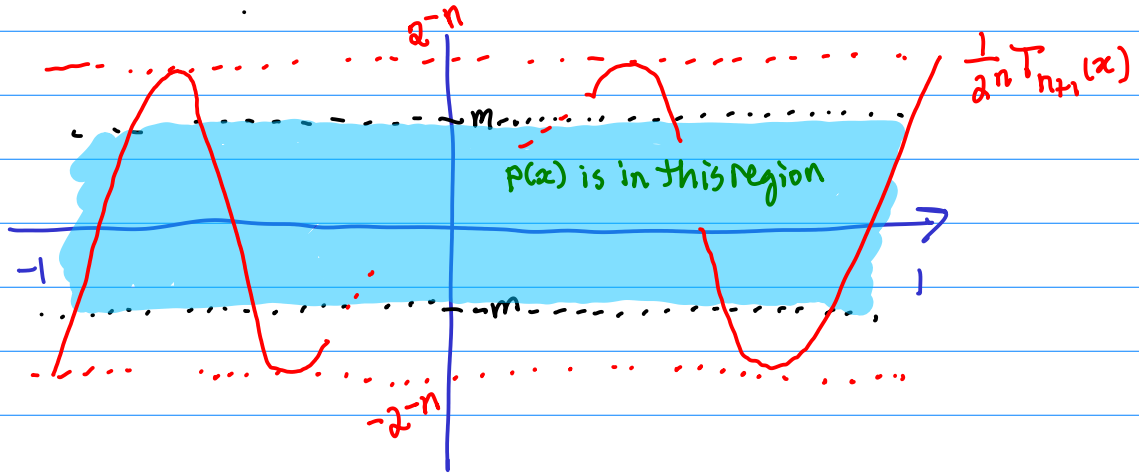
**Lemma 3.2.** *The maximum absolute value of* $\prod_{i=0}^{n}(x - x_i)$ *on* $[-1, 1]$ *is minimal if it is the normalized* $(n+1)^{th}$ *Chebyshev polynomial, i.e.,* $\prod_{i=0}^{n}(x - x_i) = 2^{-n}T_{n+1}(x)$. *The maximum absolute value is then* $2^{-n}$.

If you take any other monic polynomial $p(x)$ of degree $n+1$

leading term is just $x^{n+1}$

then $\max\{|p(x)| : x \in [-1,1]\} \geq 2^{-n} = \max\{|T_{n+1}(x)| : x \in [-1,1]\}$.

Explain why next time... suppose not then there is a polynomial $p$ such that $m = \max\{|p(x)| : x \in [-1,1]\} < 2^{-n}$.




Since the polynomial $\frac{1}{2^n}T_{n+1}$ crosses in and out of the blue region $n+2$ times, then the difference $p(x) - \frac{1}{2^n}T_{n+1}(x)$ must have $n+1$ zeros over the interval $[-1, 1]$. However, since both $p$ and $\frac{1}{2^n}T_{n+1}$ are monic polynomials of degree $n+1$ then their difference is a polynomial of degree at most $n$. Thus, there can be at most $n$ zeros total. This contradiction shows

$$\max\{|p(x)| : x \in [-1,1]\} \geq 2^{-n}.$$