Chebyshev minimize the $q$ in the error term of the interpolating polynomial theorem to find an optimally bounded interpolating polynomial!

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \underbrace{\prod_{i=0}^{n}(x - x_i)}_{q(x)}.$$

$$|f(x) - p(x)| \leq \frac{1}{(n+1)!} |f^{(n+1)}(\xi)| \, |q(x)|$$

minimized this term

$$\leq \frac{1}{(n+1)!} \max\{|f^{(n+1)}(z)| : z \in [-1,1]\} \max\{|q(z)| : z \in [-1,1]\}$$

this is a rough upper bound.

This may not be the best polynomial approximation. It's just the best bound on that approximation by the interpolating polynomial theorem.

Prefer to find the polynomial that directly minimizes

$$\max\{|f(x) - p(x)| : x \in [-1,1]\}.$$

think of this as a norm measing distance

$$\|f\|_\infty = \max\{|f(x)| : x \in [-1,1]\}$$

In analogy to linear algebra. For $x \in \mathbb{R}^n$ then

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x \cdot x}$$

$x \cdot y = \sum_{i=1}^{n} x_i y_i$

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} \quad \text{for any } p \geq 1.$$

explains why the notation is there.

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

For functions we have

satisfies properties of norms...

$$\|f\|_2 = \sqrt{\int_{-1}^{1} (f(x))^2 \, dx} = \sqrt{\langle f, f \rangle}$$

$$\|f\|_p = \sqrt[p]{\int_{-1}^{1} |f(x)|^p \, dx}$$

where $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$

Now, we want to find the polynomial that minimizes $\|f - p\|$ (same norm)

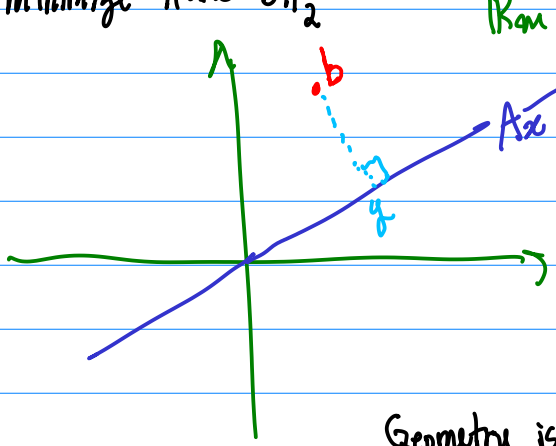
We'll minimize the root-mean-square error in the approximation

$$\text{RMSE} = \sqrt{\frac{1}{2} \int_{-1}^{1} |f(x) - p(x)|^2 \, dx} = \frac{1}{\sqrt{2}} \|f - p\|_2$$

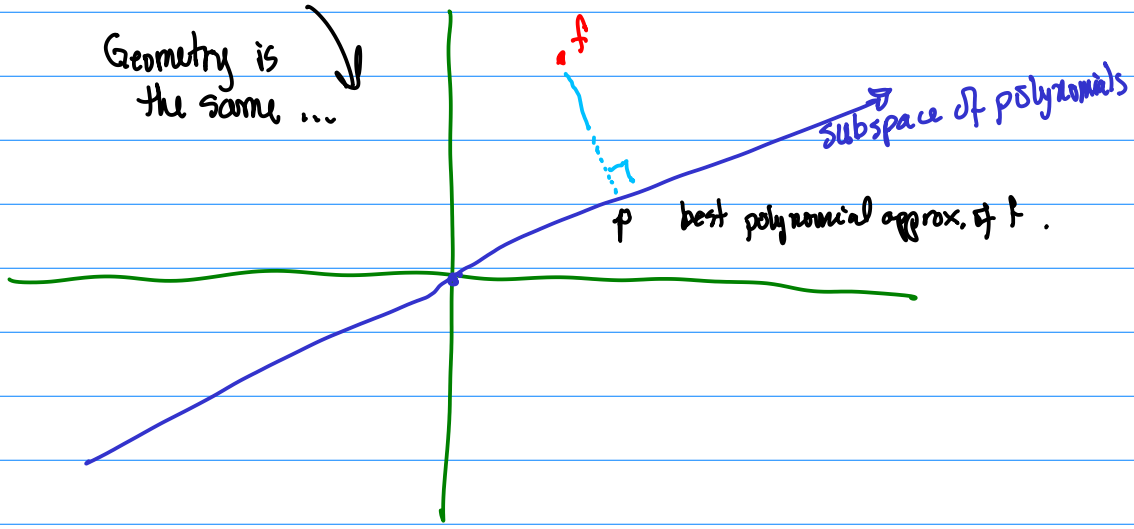
($f(x) - p(x)$: error)

↑ find polynomial that minimizes this.

minimize $\|Ax - b\|_2$




Geometry is the same ...



Gram-Schmidt to solve least squares problem.

Start with a basis $\{u_1, u_2, \ldots, u_n\}$ and obtain an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ as follows

$t_1 = u_1$ $\qquad v_1 = \frac{t_1}{\|t_1\|_2}$

$t_2 = u_2 - (v_1 \cdot u_2) v_1$ $\qquad v_2 = t_2 / \|t_2\|_2$

$t_3 = u_3 - (v_1 \cdot u_3) v_1 - (v_2 \cdot u_3) v_2$ $\qquad v_3 = t_3 / \|t_3\|_2$

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We want to create an orthonormal polynomial basis

$\{1, x, x^2, \ldots, x^n\}$ and obtain $\{v_1, v_2, \ldots, v_n\}$

$t_1 = 1$ $\qquad v_1 = \frac{t_1}{\|t_1\|_2}$

$t_2 = x - \langle v_1, x \rangle v_1$ $\qquad v_2 = t_2 / \|t_2\|_2$

$t_3 = x^2 - \langle v_1, x^2 \rangle v_1 - \langle v_2, x^2 \rangle v_2$ $\qquad v_3 = t_3 / \|t_3\|_2$

$t_4 = x^3 - \langle v_1, x^3 \rangle v_1 - \langle v_2, x^3 \rangle v_2 - \langle v_3, x^3 \rangle v_3$ $\qquad v_4 = t_4 / \|t_4\|_2$

---

Once more

skip normalization

$p_0 = 1$

$p_1 = x - \left\langle \frac{p_0}{\|p_0\|}, x \right\rangle \frac{p_0}{\|p_0\|}$

$p_2 = x^2 - \left\langle \frac{p_0}{\|p_0\|}, x^2 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^2 \right\rangle \frac{p_1}{\|p_1\|}$

$$p_3 = x^3 - \left\langle \frac{p_0}{\|p_0\|}, x^3 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^3 \right\rangle \frac{p_1}{\|p_1\|} - \left\langle \frac{p_2}{\|p_2\|}, x^3 \right\rangle \frac{p_2}{\|p_2\|}$$

deg 0 deg 1 deg 2

Thus these polynomials are orthogonal so

$$\langle p_i, p_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \|p_i\|^2 & \text{if } i = j \end{cases}$$

They are not unit norm but they are monic polynomials.

We make a simplified set of equations using a recursively defined starting basis instead of $\{1, x, x^2, x^3, \dots\}$. Next time...