Gram-Schmidt to orthogonalize $\{1, x, x^2, \dots x^n\}$ to $\{p_0, p_1, p_2 \dots, p_n\}$

$p_0 = 1$

$p_1 = x - \left\langle \frac{p_0}{\|p_0\|}, x \right\rangle \frac{p_0}{\|p_0\|}$

$p_2 = x^2 - \left\langle \frac{p_0}{\|p_0\|}, x^2 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^2 \right\rangle \frac{p_1}{\|p_1\|}$

$p_3 = x^3 - \left\langle \frac{p_0}{\|p_0\|}, x^3 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^3 \right\rangle \frac{p_1}{\|p_1\|} - \left\langle \frac{p_2}{\|p_2\|}, x^3 \right\rangle \frac{p_2}{\|p_2\|}$

deg 0 deg 1 deg 2

Note that:

$$\langle p_i, p_j \rangle = \int_{-1}^{1} p_i(x)\, p_j(x)\, dx = 0 \quad \text{if} \quad \boxed{i \neq j}.$$

If q is a polynomial of degree m then

$$q(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x$$

There also exists c_i's such that

$$q(x) = c_0 p_0 + c_1 p_1 + c_2 p_2 + \dots + c_m p_m$$

← can figure out what the c_i's are by orthogonality

$$\langle p_k, q \rangle = \langle p_k, c_0 p_0 + c_1 p_1 + c_2 p_2 + \dots + c_m p_m \rangle$$

$$= c_0 \langle p_k, p_0 \rangle + c_1 \langle p_k, p_1 \rangle + c_2 \langle p_k, p_2 \rangle + \dots + c_m \langle p_k, p_m \rangle$$

the only one that survives is $\langle p_k, p_k \rangle$ because of the orthogonality

$$= c_k \langle p_k, p_k \rangle \quad \text{for} \quad k = 0, 1, 2, \dots, m$$

Thus $c_k = \dfrac{\langle p_k, q \rangle}{\langle p_k, p_k \rangle}$.

$$p_0 = 1$$

$$p_1 = x - \left\langle \frac{p_0}{\|p_0\|}, x \right\rangle \frac{p_0}{\|p_0\|}$$

$$p_2 = x^2 - \left\langle \frac{p_0}{\|p_0\|}, x^2 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^2 \right\rangle \frac{p_1}{\|p_1\|}$$

$$p_3 = x^3 - \left\langle \frac{p_0}{\|p_0\|}, x^3 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^3 \right\rangle \frac{p_1}{\|p_1\|} - \left\langle \frac{p_2}{\|p_2\|}, x^3 \right\rangle \frac{p_2}{\|p_2\|}$$

monic; deg 0; deg 1; deg 2

$$p_0 = 1$$

$$p_1 = x p_0 - \left\langle \frac{p_0}{\|p_0\|}, x p_0 \right\rangle \frac{p_0}{\|p_0\|}$$

$$p_2 = x p_1 - \ ?$$

Note p_{n-1} is a monic polynomial of degree $n-1$ by definition

$x p_{n-1}$ is a monic polynomial of degree n by definition

$x p_{n-1} - x^n$ is a polynomial of degree $n-1$

↖ cancels the leading term...

$$x p_{n-1} - x^n = c_0 p_0 + c_1 p_1 + \cdots + c_{n-1} p_{n-1}$$

$$= \sum_{k=0}^{n-1} c_k p_k = \sum_{k=0}^{n-1} \frac{\langle p_k, x p_{n-1} - x^n \rangle}{\langle p_k, p_k \rangle} p_k$$

$$p_3 = x^3 - \left\langle \frac{p_0}{\|p_0\|}, x^3 \right\rangle \frac{p_0}{\|p_0\|} - \left\langle \frac{p_1}{\|p_1\|}, x^3 \right\rangle \frac{p_1}{\|p_1\|} - \left\langle \frac{p_2}{\|p_2\|}, x^3 \right\rangle \frac{p_2}{\|p_2\|}$$

deg 0; deg 1; deg 2

$$p_{n+1} = x^{n+1} - \left\langle \frac{p_0}{\|p_0\|}, x^{n+1} \right\rangle \frac{p_0}{\|p_0\|} - \cdots - \left\langle \frac{p_n}{\|p_n\|}, x^{n+1} \right\rangle \frac{p_n}{\|p_n\|}$$

$$= x^{n+1} - \sum_{k=0}^{n} \left\langle \frac{p_k}{\|p_k\|}, x^{n+1} \right\rangle \frac{p_k}{\|p_k\|}$$

recall that

$$x p_{n-1} - x^n = \sum_{k=0}^{n-1} \frac{\langle p_k, x p_{n-1} - x^n \rangle}{\langle p_k, p_k \rangle} p_k$$

$$x p_n - x^{n+1} = \sum_{k=0}^{n} \frac{\langle p_k, x p_n - x^{n+1} \rangle}{\langle p_k, p_k \rangle} p_k$$

$$p_{n+1} = \left(x^{n+1} - x p_n\right) + x p_n - \sum_{k=0}^{n} \left\langle \frac{p_k}{\|p_k\|}, x^{n+1} \right\rangle \frac{p_k}{\|p_k\|}$$

$$= -\sum_{k=0}^{n} \frac{\langle p_k, x p_n - x^{n+1} \rangle}{\langle p_k, p_k \rangle} p_k + x p_n - \sum_{k=0}^{n} \left\langle \frac{p_k}{\|p_k\|}, x^{n+1} \right\rangle \frac{p_k}{\|p_k\|}$$

$$p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle p_k, x p_n \rangle}{\langle p_k, p_k \rangle} p_k$$

This is the same as doing Gram-Schmidt with the starting basis $\{p_0, x p_0, x p_1, x p_2, \ldots, x p_{n-1}\}$ where $p_0 = 1$

instead if $\{1, x, x^2, x^3, \ldots, x^n\}$.

$p_0 = 1$

$$p_1 = x p_0 - \frac{\langle p_0, x p_0 \rangle}{\langle p_0, p_0 \rangle} p_0$$

Same

$p_0 = 1$

$$p_1 = x p_0 - \left\langle \frac{p_0}{\|p_0\|}, x p_0 \right\rangle \frac{p_0}{\|p_0\|}$$

$p_2 = x p_1 - ?$

$$p_2 = x p_1 - \frac{\langle p_0, x p_1 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\langle p_1, x p_1 \rangle}{\langle p_1, p_1 \rangle} p_1$$

$$\vdots$$

$$p_{n+1} = x p_n - \sum_{k=0}^{n} \frac{\langle p_k, x p_n \rangle}{\langle p_k, p_k \rangle} p_k$$

$$\langle p_k, x p_n \rangle = \int_{-1}^{1} p_k(x)\, x\, p_n(x)\, dx = \int_{-1}^{1} x\, p_k(x)\, p_n(x)\, dx = \langle x p_k, p_n \rangle$$

poly of degree $k+1$

Claim p_n is orthogonal to any polynomial of degree $m < n$.

$$\langle c_0 p_0 + c_1 p_1 + c_2 p_2 + \cdots + c_m p_m, p_n \rangle$$

$$= c_0 \langle p_0, p_n \rangle + c_1 \langle p_1, p_n \rangle + \cdots + c_m \langle p_m, p_n \rangle = 0$$

no $\langle p_n, p_n \rangle$ term. here because $m < n$

Therefore

$$\langle p_k, x p_n \rangle = 0 \quad \text{for} \quad k+1 < n \text{ or } k < n-1$$

$$\vdots$$

$$p_{n+1} = x p_n - \sum_{k=n-1}^{n} \frac{\langle p_k, x p_n \rangle}{\langle p_k, p_k \rangle} p_k$$

simplify

$$p_{n+1} = x p_n - \frac{\langle p_{n-1}, x p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1} - \frac{\langle p_n, x p_n \rangle}{\langle p_n, p_n \rangle} p_n$$

Simplify

$$\langle p_{n-1}, x p_n \rangle = \langle x p_{n-1}, p_n \rangle = \langle x p_{n-1} - p_n + p_n, p_n \rangle$$

$$= \langle x p_{n-1} - p_n, p_n \rangle + \langle p_n, p_n \rangle = \langle p_n, p_n \rangle$$

poly of deg. $n-1$

$$p_{n+1} = x p_n - \underbrace{\frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle}}_{\beta_n} p_{n-1} - \underbrace{\frac{\langle p_n, x p_n \rangle}{\langle p_n, p_n \rangle}}_{\alpha_n} p_n$$

$$p_{n+1} = (x - \alpha_n) p_n - \beta_n p_{n-1}$$

can figure out what α_n and β_n are...

Question: Why is $\alpha_n = 0$ ...

$p_0 = 1$

$p_1 = x$

$p_2 = x^2 - \frac{1}{3}$

$p_3 = x^3 - \frac{3}{5}x$

⋮

- pattern of even or odd continues. That means $(p_k(x))^2$ is even ...

$$\langle p_n, x p_n \rangle = \int_{-1}^{1} \underbrace{(p_n(x))^2}_{\text{even}} \underbrace{x}_{\text{odd}} \, dx = 0$$

so $\alpha_n = 0$ ...