Fixed point iterations: 1-step method.

Note $x_n = x^{(n)}$ in the text.

$$x_{n+1} = g(x_n)$$

assume g is a continuous function

Assume it converges $x_n \to r$ as $n \to \infty$

$$r = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(r)$$

since g is continuous

Therefore if x_n converges and g is continuous then the limit is a fixed point of g. In other words $g(r) = r$.

Simple example finding $\sqrt{2}$. Root of the equation $x^2 - 2 = 0$

Idea: Introduce a shift in x...

$$(x-1)^2 = x^2 - 2x + 1$$

$$x^2 = (x-1)^2 + 2x - 1$$

g = what's left over...

$2x - 1$: easy to invert

Thus substituting

$$(x-1)^2 + 2x - 3 = 0$$

solve for this x

$$2x = 3 - (x-1)^2$$

$$x = \frac{3-(x-1)^2}{2}$$

So if $g(x) = \frac{3-(x-1)^2}{2}$ then $\sqrt{2}$ is a fixed point of g.

In linear algebra we wrote $A = M - N$ where M was easy to invert and then made the iteration

$$Mx^{(k+1)} = Nx^{(k)} + b$$ solved for $x^{(k+1)}$ in terms of $x^{(k)}$

$$g(x) = \frac{3-(x-1)^2}{2}$$

$x_0 = 1$

```
julia> g(x)=(3-(x-1)^2)/2
g (generic function with 1

julia> x0=1.0
1.0

julia> x1=g(x0)
1.5

julia> x2=g(x1)
1.375

julia> x3=g(x2)
1.4296875

julia> sqrt(2)
1.4142135623730951
```

```
julia> sqrt(2)
1.4142135623730951

julia> xn=x0
       for n=1:50; xn=g(xn); end

julia> xn
1.414213562373095
```

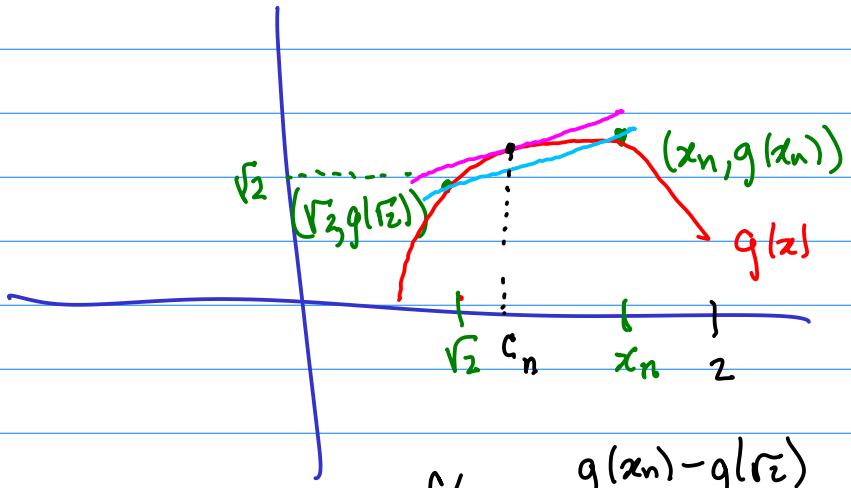
← really converged to $\sqrt{2}$

Let's assume g is continuously differentiable (derivative is continuous).

$e_n = x_n - \sqrt{2}$ $\qquad x_{n+1} = g(x_n)$ $\qquad \sqrt{2} = g(\sqrt{2})$

$$e_{n+1} = x_{n+1} - \sqrt{2} = g(x_n) - g(\sqrt{2})$$

Mean value theorem.



$$f'(c_n) = \frac{g(x_n) - g(\sqrt{2})}{x_n - \sqrt{2}}$$

$$e_{n+1} = x_{n+1} - \sqrt{2} = g(x_n) - g(\sqrt{2}) = f'(c_n)(x_n - \sqrt{2}) = f'(c_n)\, e_n$$

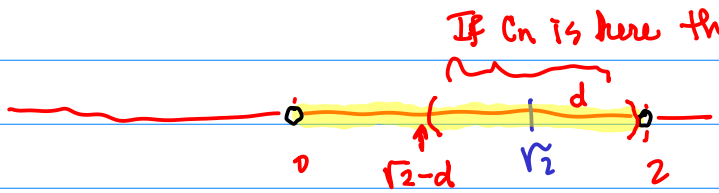
Note if $|f'(c_n)| < 1$ then the error gets smaller...

$$g(x) = \frac{3 - (x-1)^2}{2} = \frac{3}{2} - \frac{1}{2}(x-1)^2$$

$$g'(x) = -(x-1) = 1 - x$$

For what values of c is $|g'(c)| < 1$

$$|x - 1| < 1 \quad \text{for} \quad x \in (0, 2)$$



Since $|x_{n+1} - \sqrt{2}| < |x_n - \sqrt{2}|$ then if x_n is in the interval so will x_{n+1} be in the interval...

$d = 2-\sqrt{2}$ $\qquad \sqrt{2}-d = \sqrt{2}-(2-\sqrt{2}) = 2\sqrt{2}-2$

So if $x_0 \in (2\sqrt{2}-2, 2)$ then $x_n \to \sqrt{2}$.

$1 \in (0.82843, 2)$

So as $x_n \to \sqrt{2}$ then c_n which is between x_n and $\sqrt{2}$ also converges $c_n \to \sqrt{2}$ as $n \to \infty$.

$$g'(c_n) = c_n - 1 \to \sqrt{2} - 1 \approx 0.41421356\ldots$$

So once c_n is close to 0.41421356 then

$$|e_{n+1}| \leq 0.4143\,|e_n| \quad \text{for all subsequent } n.$$

The rate of convergence is a little better than the bisection method. It decreases by a little more than half each iteration. Note that this is linear convergence

$p=1$ $\qquad \lim_{n\to\infty} \dfrac{|e_{n+1}|}{|e_n|^1} = \sqrt{2}-1$ so linear convergence

Try optimizing the shift

$$(x-m)^2 = x^2 - 2mx + m^2 \qquad x^2 = (x-m)^2 + 2mx - m^2$$

easy to invert.

$$x^2 - 2 = 0$$

$$(x-m)^2 + 2mx - m^2 - 2 = 0$$

$$2mx = m^2 + 2 - (x-m)^2$$

$$x = \frac{m^2 + 2 - (x-m)^2}{2m} \qquad g(x) = \frac{m^2 + 2 - (x-m)^2}{2m}$$

Now we optimize so that $g'(c_n)$ is small ...

$$g(x) = \frac{m^2 + 2 - (x-m)^2}{2m} = \frac{m^2 + 2}{2m} - \frac{(x-m)^2}{2m}$$

$$g'(x) = -\frac{x-m}{m} = \frac{m-x}{m}$$

$$|g'(c)| = \left|\frac{m-c}{m}\right|$$

I know c is close to $\sqrt{2}$ so I need m close to $\sqrt{2}$

Ideally just take $m = \sqrt{2}$. But that's what we don't know and are trying to approximate...

Since x_n are approximating $\sqrt{2}$ take $m = x_n$ at each iteration...

$$g_n(x) = \frac{x_n^2 + 2 - (x-x_n)^2}{2x_n}$$

New iteration is $x_{n+1} = g_n(x_n)$

$$g(x_n) = \frac{x_n^2 + 2 - (x_n - x_n)^2}{2x_n} = \frac{x_n^2 + 2}{2x_n}$$

Thus, $g(x) = \frac{x^2+2}{2x}$ ← this is newtons Method.

Newton $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $f(x) = x^2 - 2$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

So I have Newton's method.