Math 466/666: Sample Final Exam Version E

1. For $A \in \mathbf{R}^{n\times n}$ the matrix norm induced by the ℓ^1 vector norm is

$$\|A\|_1 = \max\Big\{ \sum_{i=1}^{n} |a_{ij}| : j = 1, \ldots, n \Big\}$$

and the matrix norm induced by the ℓ^∞ vector norm is

$$\|A\|_\infty = \max\Big\{ \sum_{j=1}^{n} |a_{ij}| : i = 1, \ldots, n \Big\}.$$

Suppose

$$A = \begin{bmatrix} 2 & -2 & 0 & -2 \\ 3 & -1 & -2 & -4 \\ -4 & -4 & -4 & 2 \\ -1 & -1 & -2 & 2 \end{bmatrix}.$$

(i) Find $\|A\|_1$

(ii) Find $\|A\|_\infty$

2. For $v \in \mathbf{R}^n$ show that $\|v\|_2 \le \sqrt{n}\,\|n\|_\infty$.

3. Suppose that the function $g\colon \mathbf{R}^n \to \mathbf{R}^n$ is a contraction in the ∞-norm on the closed subset D of $\mathbf{R}^n$. Thus, for some $L \in (0,1)$ holds

$$\|g(x) - g(y)\|_\infty \le L\|x - y\|_\infty \qquad \text{for all} \qquad x, y \in D.$$

Use the fact that

$$\|g(x) - g(y)\|_p \le n^{1/p}\|g(x) - g(y)\|_\infty$$

to show g is a contraction in the p-norm provided $L < n^{-1/p}$.

4. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ has $n+1$ continuous derivatives. Prove that

$$f(x) = \sum_{k=0^n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + R_n(x)$$

where

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad \text{for some } c \text{ between } x \text{ and } x_0.$$

5. Please fill in the missing blanks to complete the following definition.

The Matrix Norm. Given any norm $\|\cdot\|$ on the space $\mathbf{R}^n$ of n-dimensional vectors with real entries, the subordinate matrix norm on the space $\mathbf{R}^{n\times n}$ of $n \times n$ matrices is defined by

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6. State the bisection method for approximating a solution to $f(x) = 0$.

7. What is partial pivoting, when is it used and what is the purpose of partial pivoting?

8. Suppose $A = PLU$ where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 1 & -5 \\ 0 & -2 & 8 \\ 0 & 0 & -7 \end{bmatrix}.$$

Use this factorization to find $\det(A)$.

9. Let $R^{(pq)}(\varphi)$ be the matrix whose elements r_{ij} are the same as the identity except for the four elements

$$\begin{aligned} r_{pp} &= \cos\varphi, & r_{pq} &= \sin\varphi \\ r_{qp} &= -\sin\varphi, & r_{qq} &= \cos\varphi. \end{aligned}$$

Let $A \in \mathbf{R}^{n\times n}$ be symmetric. Set $A^{(0)} = A$ and consider the iteration

$$A^{(k+1)} = R^{(pq)}(\varphi_k)^T A^{(k)} R^{(pq)}(\varphi_k)$$

with p and q chosen such that $|A_{pq}^{(k)}| = \max\{\, |A_{ij}^{(k)}| : i \neq j \,\}$ and φ_k such that

$$(A_{pp}^{(k)} - A_{qq}^{(k)}) \cos\varphi_k \sin\varphi_k + A_{pq}^{(k)}(\cos^2\varphi_k - \sin^2\varphi_k) = 0.$$

What is the idea behind this iteration and to what will $A^{(k)}$ converge?

10. Let $H = I - 2vv^T$ where $v \in \mathbf{R}^n$ is a unit vector.

(i) Show that $H^T = H$.

(ii) Show that $H^2 = I$.

11. Let $A \in \mathbf{R}^3$ and suppose the eigenvalues of the matrix $B = A^T A$ are given by $\lambda_1 = 1$, $\lambda_2 = 7$ and $\lambda_3 = 9$. Use this information to find $\|A\|_2$.

12. Recall the Gerschgorin theory given by

Theorem. Let $n \geq 2$ and $A \in \mathbf{C}^{n\times n}$. All eigenvalues of the matrix A lie in the region $D = \bigcup_{i=1}^{n} D_i$, where

$$D_i = \{\, z \in \mathbf{C} : |z - a_{ii}| \leq R_i \,\} \qquad \text{where} \qquad R_i = \sum_{j\neq i} |a_{ij}|$$

are the Gerschgorin disks.

Theorem. Suppose $1 \leq p \leq n-1$ and the Gerschgorin discs of the matrix A can be divided into two disjoint subsets $D^{(p)}$ and $D^{(q)}$, containing p and $q = n-p$ discs respectively. Then, the union of the disks in $D^{(p)}$ contains p of the eigenvalues, and the union of the discs in $D^{(q)}$ contains $n-p$ eigenvalues. In particular, if one disc is disjoint from all the others, it contains exactly one eigenvalue, and if all the discs are disjoint then each disc contains exactly one eigenvalue.

(i) Let

$$A = \begin{bmatrix} 1.0 & 0.1 & 0.3 & 0.1 \\ -0.1 & 2.0 & -0.2 & 0.0 \\ 0.4 & -0.1 & 5.0 & -0.1 \\ -0.2 & 0.2 & -0.2 & 8.0 \end{bmatrix}$$

Find the four Gerschgorin disks D_i and their radii R_i for $i = 1, \ldots, 4$.

(ii) Which of the Gerschgorin disks are disjoint from all the others?

(iii) Let λ_i be the eigenvalues of A ordered such that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. Since A is approximately diagonal it seems reasonable to suppose $\lambda_i \approx a_{ii}$. Use the Gerschgorin theory to find a bound ρ such that $|\lambda_4 - a_{44}| \leq \rho$.

13. Suppose $A = QR$ where

$$Q = \begin{bmatrix} \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{2}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} \sqrt{6} & \sqrt{2} \\ 0 & 1 \end{bmatrix}.$$

Explain how to use this factorization to minimize $\|Ax - b\|$ and then find the minimizing value of x corresponding to $b = (2, 0, 1)$.