1. The first computer program ever written was by Ada Lovelace who wrote a program for the Analytical Engine to compute Bernoulli numbers. The Bernoulli number $B_n$ is given by $B_n = B_n(0)$ where $B_n(x)$ is the unique polynomial of degree $n$ such that
\[ \int_x^{x+1} B_n(t) dt = x^n. \]

Find $B_n$ for $n = 0, 1, 2$ and $3$ by substituting a polynomial of degree $n$ into the integral and solving for the coefficients so that equality holds. You may use Maple or some other computer algebra system or do the calculation by hand.

Writing $B_0(x) = \alpha$ for some constant $\alpha$ and substituting into the equation yields
\[ \int_x^{x+1} \alpha dt = \alpha = 1 \]
Consequently $\alpha = 1$ implies $B_0 = B_0(0) = 1$.

Writing $B_1(x) = \alpha + \beta x$ for constants $\alpha$ and $\beta$ yields
\[ \int_x^{x+1} (\alpha + \beta t) dt = \alpha + \frac{\beta}{2} ((x+1)^2 - x^2) = \alpha + \frac{\beta}{2} (2x+1) \]
\[ = \left( \alpha + \frac{\beta}{2} \right) + \beta x = x. \]
Consequently
\[ \alpha + \frac{\beta}{2} = 0 \quad \text{and} \quad \beta = 1. \]
Therefore $\alpha = -1/2$ and $B_1(x) = -1/2 + x$ imply $B_1 = B_1(0) = -1/2$.

Writing $B_2(x) = \alpha + \beta x + \gamma x^2$ for constants $\alpha$, $\beta$ and $\gamma$ yields
\[ \int_x^{x+1} (\alpha + \beta t + \gamma t^2) dt = \left( \alpha + \frac{\beta}{2} \right) + \beta x + \frac{\gamma}{3} ((x+1)^3 - x^3) \]
\[ = \left( \alpha + \frac{\beta}{2} \right) + \beta x + \frac{\gamma}{3} (3x^2 + 3x + 1) \]
\[ = \left( \alpha + \frac{\beta}{2} + \frac{\gamma}{3} \right) + (\beta + \gamma) x + \gamma x^2 = x^2. \]
Consequently
\[ \alpha + \frac{\beta}{2} + \frac{\gamma}{3} = 0, \quad \beta + \gamma = 0 \quad \text{and} \quad \gamma = 1. \]
Therefore $\beta = -1$, $\alpha = 1/6$ and $B_2(x) = 1/6 - x + x^2$ implies $B_2 = B_2(0) = 1/6$. 
Writing $B_3(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$ for constants $\alpha$, $\beta$, $\gamma$ and $\delta$ yields

$$\int_x^{x+1} (\alpha + \beta t + \gamma t^2 + \delta t^3) dt$$

$$= \left( \alpha + \frac{\beta}{2} + \frac{\gamma}{3} \right) + (\beta + \gamma)x + \gamma x^2 + \delta \frac{x^4}{4} - x^4$$

$$= \left( \alpha + \frac{\beta}{2} + \frac{\gamma}{3} \right) + (\beta + \gamma)x + \gamma x^2 + \frac{\delta}{4}(4x^3 + 6x^2 + 4x + 1)$$

$$= \left( \alpha + \frac{\beta}{2} + \frac{\gamma}{3} + \frac{\delta}{4} \right) + (\beta + \gamma + \delta)x + \left( \gamma + \frac{3}{2}\delta \right)x^2 + \delta x^3 = x^3.$$ 

Consequently

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{3} + \frac{\delta}{4} = 0, \quad \beta + \gamma + \delta = 0, \quad \gamma + \frac{3}{2}\delta = 0 \quad \text{and} \quad \delta = 1.$$ 

Therefore

$$\gamma = -\frac{3}{2}, \quad \beta = \frac{1}{2}, \quad \alpha = -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0$$

and $B_3(x) = (1/2)x - (3/2)x^2 + x^3$ implies $B_3 = B_3(0) = 0.$
2. By the Fundamental Theorem of Calculus it follows that

\[
\frac{d}{dx} \int_x^{x+1} B_n(t) dt = B_n(x+1) - B_n(x) = \int_x^{x+1} B'_n(t) dt.
\]

Use this fact to show that \( B'_n(x) = nB_{n-1}(x) \).

By definition

\[
\int_x^{x+1} B_n(t) dt = x^n
\]

Therefore

\[
\frac{d}{dx} \int_x^{x+1} B_n(t) dt = nx^{n-1}.
\]

Since, as noted above

\[
\int_x^{x+1} B'_n(t) dt = \frac{d}{dx} \int_x^{x+1} B_n(t) dt
\]

it follows upon dividing by \( n \) that

\[
\int_x^{x+1} n^{-1} B'_n(t) dt = x^{n-1}.
\]

Since \( B_n(t) \) is a polynomial of degree \( n \), then \( n^{-1} B'_n(t) \) is a polynomial of degree \( n - 1 \). Moreover, by definition \( B_{n-1}(x) \) is the unique polynomial of degree \( n - 1 \) such that

\[
\int_x^{x+1} B_{n-1}(t) dt = x^{n-1}.
\]

Therefore, we conclude \( n^{-1} B'_n(x) = B_{n-1}(x) \) or, in otherwords, that \( B'_n(x) = nB_{n-1}(x) \).
3. By the Fundamental Theorem of Calculus we also have

\[ \int_0^x B'_n(t) \, dt = B_n(x) - B_n(0) \]

or equivalently

\[ B_n(x) = B_n + \int_0^x nB_{n-1}(t) \, dt. \]

Integrate the above equality in \( x \) from 0 to 1, then interchange the order of integration to obtain the relation that

\[ B_n = \int_0^1 t nB_{n-1}(t) \, dt \quad \text{for} \quad n > 1. \]

Integrating each side of the equality yields

\[ \int_0^1 B_n(x) \, dx = \int_0^1 B_n \, dx + \int_0^1 \int_0^x nB_{n-1}(t) \, dt \, dx. \tag{3.1} \]

Setting \( x = 0 \) in the defining equality

\[ \int_x^{x+1} B_n(t) \, dt = x^n \]

shows that the left side of (3.1) is exactly equal to zero. Since \( B_n \) is a constant, the first term of the right is \( B_n \). It remains to switch the order of integration in the last term. Doing so obtains

\[
\int_0^1 \int_0^x nB_{n-1}(t) \, dt \, dx = \int_0^1 \int_t^1 nB_{n-1}(t) \, dx \, dt
\]

\[ = \int_0^1 (1-t)nB_{n-1}(t) \, dt = \int_0^1 nB_{n-1}(t) \, dt - \int_0^1 t nB_{n-1}(t) \, dt \]

Now, if \( n > 1 \) we may set \( x = 0 \) in the defining equality

\[ \int_x^{x+1} B_{n-1}(t) = x^{n-1} \]

to obtain that

\[ \int_0^1 nB_{n-1}(t) \, dt = 0. \]

Note if \( n = 1 \) it is not possible to set \( x = 0 \) above because that would result in the indeterminate form \( 0^0 \). This is why the recurrence is only good for \( n > 1 \). It follows that

\[ B_n = \int_0^1 t nB_{n-1}(t) \, dt \quad \text{for} \quad n > 1. \]
4. Write $B_{n-1}(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$ and use the identity

$$B_n(x) = \int_0^1 t n B_{n-1}(t) dt + \int_0^x n B_{n-1}(t) dt$$

derived in the previous step to find formulas for $B_n$ and $B_n(x)$ in terms of the $\alpha_k$.

Integrating yields

$$\int_0^1 t n B_{n-1}(t) dt = \int_0^1 \sum_{k=0}^{n-1} n \alpha_k t^{k+1} dt = \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+2} t^{k+2} \bigg|_0^1 = \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+2}$$

and

$$\int_0^x n B_{n-1}(t) dt = \int_0^x \sum_{k=0}^{n-1} n \alpha_k t^k dt = \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+1} t^{k+1} \bigg|_0^x = \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+1} x^{k+1}.$$ 

Therefore

$$B_n = \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+2} \quad \text{and} \quad B_n(x) = B_n + \sum_{k=0}^{n-1} \frac{n \alpha_k}{k+1} x^{k+1}.$$
5. Starting with $B_1(x) = x - 1/2$, write a program that computes the Bernoulli numbers by means of the formulas derived in the previous step. Use your program to print a table listing the values of $n$ and $B_n$ for $n = 1, 2, \ldots, 10$.

The program is

```c
#include <stdio.h>
#include <math.h>

#define N 10
double alpha[N+1]={-0.5,1}; // B1(x)=x-1/2;

int main(){
    printf("# n Bn
    ");
    for(int n=1;;){
        printf(" %6d %24s
","n","Bn");
        n++;
        if(n>N) break;
        double b=0;
        for(int k=0;k<n;k++) b+=alpha[k]/(k+2);
        for(int k=n;k>0;k--) alpha[k]=n*alpha[k-1]/k;
        alpha[0]=n*b;
    }
    return 0;
}
```

and the output is

<table>
<thead>
<tr>
<th>#</th>
<th>n</th>
<th>Bn</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.16666666666667</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>-1.8882395647069e-12</td>
</tr>
<tr>
<td>10</td>
<td>0.075757575738698</td>
<td>0.075757575738698</td>
</tr>
</tbody>
</table>