1. For each of the following numbers

$$57.82156, \qquad 5.782156, \qquad 0.5782156, \qquad 0.05782156$$

(i) round to three significant digits (3S),

(ii) round to three decimal places (3D).

(i) $57.82156 \xrightarrow{\text{round to 3S}} 57.8$

$5.782156 \xrightarrow{\text{round to 3S}} 5.78$

$0.5782156 \xrightarrow{\text{round to 3S}} 0.578$

$0.05782156 \xrightarrow{\text{round to 3S}} 0.0578$

(ii) $57.82156 \xrightarrow{\text{round to 3D}} 57.822$

$5.782156 \xrightarrow{\text{round to 3D}} 5.782$

$0.5782156 \xrightarrow{\text{round to 3D}} 0.578$

$0.05782156 \xrightarrow{\text{round to 3D}} 0.058$

2. Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Let S_n^* be the approximation of S_n obtained with a hypothetical computer that uses chopping arithmetic. Is it true that $S_n^* \leq S_n$? If so explain why; if not provide a counterexample where $S_n^* > S_n$.

Since $\frac{1}{k} \geq 0$ is positive for $k = 1, \dots, n$ then chopping yields an approximation $\left(\frac{1}{k}\right)^*$ of $\frac{1}{k}$ such that $0 \leq \left(\frac{1}{k}\right)^* \leq \frac{1}{k}$.

Now procede by induction.

For $n=1$ then $S_1 = 1$ and $S_1^* = 1 \leq S_1$

Suppose $S_n^* \leq S_n$ then

$$S_{n+1}^* = \left(S_n^* + \left(\frac{1}{n+1}\right)^*\right)^* \leq S_n^* + \left(\frac{1}{n+1}\right)^* \leq S_n + \frac{1}{n+1} = S_{n+1}$$

which shows $S_n^* \leq S_n$ for all n.

3. Evaluate the following using three-digit decimal normalized floating point arithmetic with rounding:

(i) $4.56 \times 10^2 + 8.17 \times 10^2$

(ii) $4.56 \times 10^2 + 8.17 \times 10^1$

(iii) $4.56 \times 10^2 + 8.17 \times 10^{-1}$

(iv) $4.56 \times 10^2 + 8.17 \times 10^{-2}$

(i) $4.56 \times 10^2 + 8.17 \times 10^2 = 12.73 \times 10^2 = 1.273 \times 10^3 \xrightarrow{\text{round}} 1.27 \times 10^3$

(ii) $4.56 \times 10^2 + 8.17 \times 10^1 = 5.377 \times 10^2 \xrightarrow{\text{round}} 5.38 \times 10^2$

(iii) $4.56 \times 10^2 + 8.17 \times 10^{-1} \approx 4.56817 \times 10^2 \xrightarrow{\text{round}} 4.57 \times 10^2$

(iv) $4.56 \times 10^2 + 8.17 \times 10^{-1} \approx 4.560817 \times 10^2 \xrightarrow{\text{round}} 4.56 \times 10^2$

4. Let $x \in \mathbf{R}$ and x^* be the approximation of x obtained by rounding to four signifiant digits. Suppose $x^* = 4.562 \times 10^2$.

(i) Find the smallest interval that contains x.

(ii) Find a bound for the absolute error $e_{\text{abs}} = |x - x^*|$.

(i) $x^* = 4.562 \times 10^2 \pm 0.0005 \times 10^2$

round to even means $4.5625 \times 10^2 \xrightarrow{\text{round}} 4.562 \times 10^2$

and $4.5615 \times 10^2 \longrightarrow 4.562 \times 10^2$

Therefore the endpoints are included and smallest interval that contains x is

$$x \in [4.5615 \times 10^2, 4.5625 \times 10^2]$$

(ii) $e_{abs} = |x - x^*| \leq 0.0005 \times 10^2 = 5 \times 10^{-2}$

5. Estimate the accumulated errors in the results of Question 3 assuming that all values are correct to three significant digits. Use either interval arithmetic or bounds based on the fact that accumuated error is the sum of the propagated and generated errors.

(i) $4.56\times10^2 + 8.17\times10^2 = 12.73\times10^2 = 1.273\times10^3 \xrightarrow{\text{round}} 1.27\times10^3$

$e_{initial}(4.56\times10^2) \lesssim 0.005\times10^2$

$e_{initial}(8.17\times10^2) \leq 0.005\times10^2$

$e_{prop} \lesssim 0.005\times10^2 + 0.005\times10^2 = 0.01\times10^2 = 1.$

$e_{gen} = |1.273\times10^3 - 1.27\times10^3| \approx 3$

Therefore the accumulated error satisfies

$e_{acc} \leq e_{prop} + e_{gen} \lesssim 4.$

(ii) $4.56\times10^2 + 8.17\times10^1 = 5.377\times10^2 \xrightarrow{\text{round}} 5.38\times10^2$

$e_{initial}(8.17\times10^1) \leq 0.005\times10^1$

$e_{prop} \lesssim 0.005\times10^2 + 0.005\times10^1 = 5.5\times10^{-1}$

$e_{gen} = |5.377\times10^2 - 5.38\times10^2| \approx 3\times10^{-1}$

Therefore

$e_{acc} \leq e_{prop} + e_{gen} \lesssim 5.5\times10^{-1} + 3\times10^{-1} = 8.5\times10^{-1}$

(iii) $4.56\times10^2 + 8.17\times10^{-1} \approx 4.56817\times10^2 \xrightarrow{\text{round}} 4.57\times10^2$

$e_{initial}(8.17\times10^{-1}) \leq 0.005\times10^{-1}$

$e_{prop} \lesssim 0.005\times10^2 + 0.005\times10^{-1} = 5.005\times10^{-1}$

$e_{gen} = |4.56817\times10^2 - 4.57\times10^2| \approx 1.83\times10^{-1}$

Therefore

$e_{acc} \leq e_{prop} + e_{gen} \lesssim 5.005\times10^{-1} + 1.83\times10^{-1} = 6.835\times10^{-1}$

(iv) $4.56 \times 10^{2} + 8.17 \times 10^{-1} \approx 4.560817 \times 10^{2} \xrightarrow{\text{round}} 4.56 \times 10^{2}$

$e_{initial}(8.17 \times 10^{-2}) \leq 0.005 \times 10^{-2}$

$e_{prop} \lesssim 0.005 \times 10^{2} + 0.005 \times 10^{-2} = 5.0005 \times 10^{-1}$

$e_{gen} = \left| 4.560817 \times 10^{2} - 4.56 \times 10^{2} \right| \approx 8.17 \times 10^{-2}$

Therefore

$e_{acc} \leq e_{prop} + e_{gen} \lesssim 5.0005 \times 10^{-1} + 8.17 \times 10^{-2} = 5.8175 \times 10^{-1}$

6. Find the Taylor series expansion about $x = 0$ for each of the following functions:

(i) $\sin x$

(ii) $\sqrt{1-x}$

(iii) e^{2x}

For each series determine a general remainder term.

Since $x_0 = 0$, then in all cases

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0)$$

$$R_n(x) = \int_0^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

for some c between 0 and x

(i) $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

In general

$$f^{(k)}(x) = \left\{\begin{array}{ll} \sin x & \text{if } k = 4l \\ \cos x & \text{if } k = 4l+1 \\ -\sin x & \text{if } k = 4l+2 \\ -\cos x & \text{if } k = 4l+3 \end{array}\right\} = \left\{\begin{array}{ll} (-1)^l \sin x & \text{if } k = 2l \\ (-1)^l \cos x & \text{if } k = 2l+1 \end{array}\right.$$

Thus

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \left\{\begin{array}{ll} (-1)^l \sin 0 & \text{if } k = 2l \\ (-1)^l \cos 0 & \text{if } k = 2l+1 \end{array}\right\} = \sum_{k \text{ odd}}^{n} \frac{x^k}{k!}(-1)^{(k-1)/2}$$

$$= \sum_{1 \leq 2l+1 \leq n} (-1)^l \frac{x^{2l+1}}{(2l+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

where m is chosen so $2m+1 \leq n < 2m+3$

Note that $T_n(x) = T_{n+1}(x)$ when n is odd, therefore

$$R_{2m+1}(x) = R_{2m+2}(x) = \frac{x^{2m+3}}{(2m+3)!} f^{(2m+3)}(c) = \frac{x^{2m+3}}{(2m+3)!} (-1)^{m+1} \cos(c)$$

for some c between 0 and x.

A slightly different argument appears in the lecture notes from September 20 and concludes that

$$R_{2m+1}(x) = R_{2m+2}(x) = \int_0^x \frac{(x-t)^{2m+2}}{(2m+2)!} (-1)^{m+1} \cos t \, dt$$

This is equally correct.

(ii) $f(x) = (1-x)^{1/2}$, $f'(x) = \frac{-1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{2}\cdot\frac{1}{2}(1-x)^{-3/2}$
$f'''(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1-x)^{-5/2}$, $f^{(4)}(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-7/2}$

In general

$$f^{(k)}(x) = \underbrace{-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2k-3}{2}}_{k \text{ terms}} (1-x)^{-(2k-1)/2}$$

Thus

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0) = \sum_{k=0}^{n} \frac{-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-3}{2}}{1\cdot 2\cdot 3\cdots k} x^k = \sum_{k=0}^{n} \binom{1/2}{k} (-x)^k$$

where the generalized binomial coefficients are defined as

$$\binom{\alpha}{0} = 1$$

$$\binom{\alpha}{1} = \alpha$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{1\cdot 2 \qquad \cdot k} \qquad \text{for } k \geq 2$$

The remainder is

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \binom{1/2}{n+1} x^{n+1} (1-c)^{-(2n-1)/2}$$

for some c between 0 and x

Alternatively

$$R_n(x) = \int_0^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt = \binom{1/2}{n+1}(n+1)\int_0^x (x-t)^n (1-t)^{-(2n-1)/2}\,dt$$

(iii) $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 2^2 e^{2x}$, $f'''(x) = 2^3 e^{2x}$

In general $f^{(k)}(x) = 2^k e^{2x}$

Thus, $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0) = \sum_{k=0}^{n} \frac{(2x)^k}{k!}$

and $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{x^{n+1}}{(n+1)!} 2^{n+1} e^{2c} = \frac{(2x)^{n+1}}{(n+1)!} e^{2c}$

for some c between 0 and x

Alternatively

$$R_n(x) = \int_0^x \frac{1}{n!}(x-t)^n f^{(n+1)}(t)\,dt = \int_0^x \frac{1}{n!}(x-t)^n 2^{n+1} e^{2t}\,dt.$$

7. Use the remainder term in Question 6(iii) to find the degree n of the Taylor polynomial approximation to e^{2x} that gives 4D accuracy for all x between 0 and 1.

For 4D accuracy we need $|R_n(x)| \lesssim 0.00005$. Thus we need to show

$$\left| \int_0^x \frac{1}{n!}(x-t)^n 2^{n+1} e^{2t}\, dt \right| \lesssim 0.00005$$

Since $x \in [0,1]$ thus $x \geq 0$ and $R_n(x) = \frac{(2x)^{n+1}}{(n+1)!} e^{2c}$ for some $c \in [0,x]$.

Now

$$\left| \frac{(2x)^{n+1}}{(n+1)!} e^{2c} \right| \leq \frac{|2x|^{n+1} e^{2x}}{(n+1)!} \leq \frac{2^n e^2}{(n+1)!}$$

implies choosing n so large that $\frac{2^n e^2}{(n+1)!} \lesssim 0.00005 = 5 \times 10^{-5}$ is enough to satisfy the desired error bounds. Since

n	$\frac{2^n e^2}{(n+1)!}$
1	7.389...
2	4.926...
3	2.463...
4	0.9852...
5	0.3284...
6	0.09382...
7	0.02345...
8	0.0052127...
9	0.00104...

n	$\frac{2^n e^2}{(n+1)!}$
10	0.0001895...
11	$3.159 \times 10^{-5} \lesssim 5 \times 10^{-5}$

Then the degree 11 Taylor polynomial is enough to guarantee 4D accuracy for all $x \in [0,1]$.

8. Evaluate $p(2.1)$ and $p'(2.1)$, where $p(x) = x^3 - 2x^2 + 2x + 3$, using the technique of nested multiplication.

Note that $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where

$$a_0 = 3, \quad a_1 = 2, \quad a_2 = -2 \quad \text{and} \quad a_3 = 1.$$

The recursive formula in the book says compute

$$p_0 = a_n \qquad q_0 = 0$$

$$p_k = p_{k-1} x + a_{n-k} \qquad q_k = q_{k-1} x + p_{k-1} \qquad \text{for } k = 1, \dots, n$$

Then $p(x) = p_n$ and $p'(x) = q_n$.

In our case $n = 3$ and $x = 2.1$. We obtain

k	p_k	q_k
0	1	0
1	$1(2.1) + (-2) = 0.1$	$0(2.1) + 1 = 1$
2	$0.1(2.1) + 2 = 2.21$	$1(2.1) + 0.1 = 2.2$
3	$2.21(2.1) + 3 = 7.641$	$2.2(2.1) + 2.21 = 6.83$

Therefore $p(2.1) = 7.641$ and $p'(2.1) = 6.83$.