Math 466/666: Homework 3 Version A

This homework is based on steps 11 through 16 (but not 14) from the text *First Steps in Numerical Analysis* by Hosking, Joe, Joyce and Turner. Students are encouraged to work in groups and consult resources outside of the required textbook when doing the homework for this class. Please cite any additional sources you used to complete your work.

1. Find the range of solutions for the system, assuming maxiumu errors in the constants as shown:

$$x - y = 1.4 \pm 0.02$$
$$x + y = 3.8 \pm 0.03.$$

2. How does partial pivoting contribute to a reduction of errors?

3. Consider solving the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 5 & -3 & 0 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & 6 & -1 & 0 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & -3 & 5 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 3 \\ 2 \end{bmatrix}.$$

(i) Show that $x = (1, 1, 1, 1, 1)$ is the exact solution.

(ii) Use the `lu` function from Julia's `LinearAlgebra` library to find the LU factorization of A. Explain why L and U have so many zeros.

(iii) Starting with $x_0 = (2, 3, 4, 3, 2)$ use 5 iterations of the Gauss-Seidel method to approximate the solution. Please include your code and the value of the n-th iteration x_n for $n = 1, \ldots, 5$ in your report.

4. How is the condition number of a matrix defined and how is it used as a test for ill-conditioning?

5. Consider the vector norms of a vector $x \in \mathbf{R}^n$ defined as

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \sqrt{x \cdot x} \qquad \text{and} \qquad \|x\|_\infty = \max\left\{ |x_i| : i = 1, \ldots, n \right\}.$$

Let $b = (-7, 8, -1, 2)$ and find $\|b\|_1$, $\|b\|_2$ and $\|b\|_\infty$.

6. [Extra Credit/Math 666] Is the ordering of the norms $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ from smallest to largest the same independent of the value of x? If so, explain why. If not provide a counter example.

7. Consider the matrix

$$A = \begin{bmatrix} 4 & 8 & -4 & -10 \\ 8 & 3 & -10 & -1 \\ -1 & -7 & 6 & -9 \\ -8 & 7 & 1 & -3 \end{bmatrix}.$$

Find the matrix norms $\|A\|_\infty$ and $\|A\|_1$.