Math 466/666: Homework 4 Version A

This homework is based on steps 18 through 24 from the text *First Steps in Numerical Analysis* by Hosking, Joe, Joyce and Turner. Students are encouraged to work in groups and consult resources outside of the required textbook when doing the homework for this class. Please cite any additional sources you used to complete your work.

1. Let $f(x) = \sqrt{x}$ and $x_i = x_0 + hi$ where $x_0 = 1$ and $h = 1/2$. Consider the following table of finite differences:

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1	1.0000					
		2247				
1.5	1.2247		-352			
		1895		126		
2	1.4142		-226		-59	
		1669		67		28
2.5	1.5811		-159		-31	
		1510		36		
3	1.7321		-123			
		1387				
3.5	1.8708					

(i) Use the approximation

$$f(x_i + \alpha h) \approx \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f_i$$

to show that

$$f(x) \approx p(x) \qquad \text{where} \qquad p(x) = \sum_{k=0}^{n} \binom{(x - x_i)/h}{k} \Delta^k f_i.$$

(ii) Let $n = 3$ and $i = 1$ to find the unique polynomial p of degree three such that

$$p(1.5) = 1.2247, \quad p(2) = 1.4142, \quad p(2.5) = 1.5811 \quad \text{and} \quad p(3) = 1.7321.$$

Please do not simplify or attempt to write p in the standard basis.

(iii) Are the third-order differences constant to within the expected rounding error? What does this tell you about the quality of the approximation $f(x) \approx p(x)$?

(iv) Evaluate $p(2.25)$ and compute $\left|\sqrt{2.25} - p(2.25)\right|$ to find the error.

2. The polynomial interpolation theorem states $f(t) = p(t) + E(t)$ where

$$E(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (t - x_{i+k})$$

for some ξ between $\min\{t, x_i, \ldots, x_{i+n}\}$ and $\max\{t, x_i, \ldots, x_{i+n}\}$.

(i) Let $f(x) = \sqrt{x}$ and let $B = \max\{\, |f^{(4)}(\xi)| : \xi \in [1.5, 3] \,\}$. Find B.

(ii) Consider the interpolating polynomial p of degree three such that

$$p(1.5) = 1.2247, \quad p(2) = 1.4142, \quad p(2.5) = 1.5811 \quad \text{and} \quad p(3) = 1.7321.$$

For this polynomial it follows for $t \in [1.5, 3]$ that

$$|E(t)| \le \frac{B}{(n+1)!} \Big| (t - 1.5)(t - 2)(t - 2.5)(t - 3) \Big|.$$

Evaluate this theoretical upper bound on the error $|E(2.25)|$.

(iii) How many times bigger is the theoretical upper bound on $|E(2.25)|$ compared to the actual error $\big|\sqrt{2.25} - p(2.25)\big|$ from 1(iv) of the previous problem? Is it possible the actual error could be bigger than theoretical upper bound?

(iv) [Extra Credit and Math 666] Compute the actual value of E(2.25) and solve to find ξ in the polynomial interpolation theorem. Verify that $\xi \in [1.5, 3]$.