HW1 problems 1.1, 1.4 and 1.6 due Sept 23.

1.1 The iteration defined by $x_{k+1} = \frac{1}{2}(x_k^2 + c)$, where $0 < c < 1$, has two fixed points ξ_1, ξ_2, where $0 < \xi_1 < 1 < \xi_2$. Show that

$$x_{k+1} - \xi_1 = \tfrac{1}{2}(x_k + \xi_1)(x_k - \xi_1)\,, \qquad k = 0, 1, 2, \dots\,,$$

and deduce that $\lim_{k\to\infty} x_k = \xi_1$ if $0 \le x_0 < \xi_2$. How does the iteration behave for other values of x_0?

The fixed points satisfy $\xi = \frac{1}{2}(\xi^2 + c)$ or $\xi^2 - 2\xi + c = 0$

$$\xi = \frac{2 \pm \sqrt{4-4c}}{2} = 1 \pm \sqrt{1-c}$$

Therefore $\xi_1 = 1 - \sqrt{1-c}$ and $\xi_2 = 1 + \sqrt{1-c}$.

Thus $\frac{1}{2}(x_k + \xi_1)(x_k - \xi_1) = \frac{1}{2}(x_k^2 - \xi_1^2) = \frac{1}{2}\left(x_k^2 - (1-\sqrt{1-c})^2\right)$

$$= \tfrac{1}{2}\left(x_k^2 - \left(1 - 2\sqrt{1-c} + 1 - c\right)\right) = \tfrac{1}{2}\left(x_k^2 + c - 2 + 2\sqrt{1-c}\right)$$

$$= x_{k+1} - \left(1 - \sqrt{1-c}\right) = x_{k+1} - \xi_1 \qquad \text{for } k = 0, 1, \dots$$

Let $e_k = x_k - \xi_1$. Then $e_{k+1} = \frac{1}{2}(x_k + \xi_1)\,e_k$ implies $|e_{k+1}| < |e_k|$ provided $\left|\frac{1}{2}(x_k + \xi_1)\right| < 1$. Equivalently

$$-2 < x_k + 1 - \sqrt{1-c} < 2 \qquad \text{or} \qquad -3 + \sqrt{1-c} < x_k < 1 + \sqrt{1-c}$$

Since $-3 + \sqrt{1-c} < 0$ and $1 + \sqrt{1-c} = \xi_2$ this condition is guaranteed when $0 \le x_k < \xi_2$.

Moreover, if $0 \leq x_k < \xi_2$ then $x_{k+1} = \frac{1}{2}(x_k^2 + c) \geq 0$.

Case $e_k > 0$ then $e_{k+1} < |e_k|$ implies

$$x_{k+1} - \xi_1 < x_k - \xi_1 \quad \text{and so} \quad x_{k+1} < x_k < \xi_2$$

Case $e_k < 0$ then $x_k < \xi_1$

$$x_{k+1} = \frac{1}{2}(x_k^2 + c) < \frac{1}{2}(\xi_1^2 + c) = \frac{1}{2}\left(1 - 2\sqrt{1-c} + 1 - c + c\right)$$

$$= 1 - \sqrt{1-c} < \xi_1 < \xi_2$$

By induction x_0 satisfying $0 \leq x_0 < \xi_2$ implies $0 \leq x_k < \xi_2$ for all $k \in \mathbb{N}$.

To see that $x_k \to \xi_1$ note that the above two cases combine to imply $x_k \leq \max(\xi_1, x_0)$ for all k.

Setting $\gamma = \frac{1}{2}(\max(\xi_1, x_0) + \xi_1)$ and noting

$$|e_{k+1}| \leq \gamma |e_k| \quad \text{for all } k$$

along with

$$\gamma < \frac{1}{2}(\xi_2 + \xi_1) = \frac{1}{2}(1 + \sqrt{1-c} + 1 - \sqrt{1-c}) = 1$$

Implies $|e_k| \leq \gamma^k |e_0| \to 0$ as $k \to \infty$ and so $x_k \to \xi_1$.

As for ξ_1 compute for ξ_2 as

$$\frac{1}{2}(x_k + \xi_2)(x_k - \xi_2) = \frac{1}{2}(x_k^2 - \xi_2^2) = \frac{1}{2}\left(x_k^2 - (1 + \sqrt{1-c})^2\right)$$

$$= \frac{1}{2}\left(x_k^2 - \left(1 + 2\sqrt{1-c} + 1 - c\right)\right) = \frac{1}{2}\left(x_k^2 + c - 2 - 2\sqrt{1-c}\right)$$

$$= x_{k+1} - (1 + \sqrt{1-c}) = x_{k+1} - \xi_2 \quad \text{for } k = 0, 1, \ldots$$

Thus, we also have

$$x_{k+1} - \xi_2 = \frac{1}{2}(x_k + \xi_2)(x_k - \xi_2)$$

To obtain a contraction at ξ_2 we need $\left|\frac{1}{2}(x_k + \xi_2)\right| < 1$; however, if $x_k > \xi_2$ then $\xi_2 > 1$ implies

$$\frac{1}{2}(x_k + \xi_2) > \xi_2 = 1 + \sqrt{1-c}$$

Therefore, $x_k > \xi_2$ implies

$$x_{k+1} = x_{k+1} - \xi_2 + \xi_2 \geq \frac{1}{2}(x_k + \xi_2)(x_k - \xi_2) + \xi_2$$

$$> (1 + \sqrt{1-c})(x_k - \xi_2) + \xi_2 = x_k + \sqrt{1-c}\,(x_k - \xi_2)$$

and so $x_{k+1} > x_k$ for all k. It follows that

$$|x_{k+1} - \xi_2| > (1 + \sqrt{1-c})|x_k - \xi_2|$$

and by induction if $x_0 > \xi_2$ then

$$|x_k - \xi_2| > (1 + \sqrt{1-c})^k |x_0 - \xi_2| \to \infty \quad \text{as} \quad k \to \infty.$$

Finally, if $x_0 \leq 0$ note that

$$x_1 = \frac{1}{2}(x_0^2 + c) > 0$$

after which $x_1 < \xi_2$ implies $x_k \to \xi_1$ and $x_1 > \xi_2$ implies $x_k \to \infty$.

Now

$$\frac{1}{2}(x_0^2+c) < \xi_2 = 1+\sqrt{1-c}$$

implies $\frac{1}{2}x_0^2 < 1-\frac{c}{2}+\sqrt{1-c}$

or $x_0^2 \leq 2-c+2\sqrt{1-c} = 1-c+2\sqrt{1-c}+1$

$$= (\sqrt{1-c})^2+2\sqrt{1-c}+1 = (\sqrt{1-c}+1)^2 = \xi_2^2$$

Since $x_0 < 0$ then

$-\xi_2 < x_0 < 0$ implies $x_k \to \xi_1$ as $k \to \infty$

If

$x_0 < -\xi_2$ then $x_k \to \infty$ as $k \to \infty$

In summary

$|x_0| < \xi_2$ implies $x_k \to \xi_1$ as $k \to \infty$

$|x_0| = \xi_2$ implies $x_k = \xi_2$ for $k = 1, 2, \ldots$

$|x_0| > \xi_2$ implies $x_k \to \infty$ as $k \to \infty$.

1.4 Consider the iteration

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}, \qquad k = 0, 1, 2, \ldots,$$

for the solution of $f(x) = 0$. Explain the connection with Newton's method, and show that (x_k) converges quadratically if x_0 is sufficiently close to the solution. Apply this method to the same example as in Example 1.7, $f(x) = e^x - x - 2$, and verify quadratic convergence beginning from $x_0 = 1$. Experiment with calculations beginning from $x_0 = 10$ and from $x_0 = -10$, and account for their behaviour.

The connection with Newton's method is as follows. First recall Newton's method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

The connection is based on the approximation to f'

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{for small } h$$

where $x = x_k$ and $h = f(x_k)$. Note that if x_k is close to the root then $f(x_k)$ is small.

Substitute this approximation into Newton's method to obtain

$$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}} = x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}.$$

Assume $f(\xi)=0$ and $f'(\xi)\neq 0$ and f'' is continuous.
By Taylor's theorem

$$f(x_k+f(x_k)) = f(x_k) + f(x_k)f'(x_k) + \frac{1}{2}f(x_k)^2 f''(a_k)$$

for some a_k between x_k and $x_k + f(x_k)$

Therefore

$$\frac{f(x_k+f(x_k)) - f(x_k)}{f(x_k)} = f'(x_k) + \frac{1}{2}f(x_k)f''(a_k).$$

Let $e_k = x_k - \xi$. Then

$$e_{k+1} = x_{k+1} - \xi = x_k - \xi - \frac{(f(x_k))^2}{f(x_k+f(x_k)) - f(x_k)}$$

$$= x_k - \xi - \frac{f(x_k)}{f'(x_k) + \frac{1}{2}f(x_k)f''(a_k)}$$

$$= e_k - \frac{f(x_k)}{f'(x_k)} \cdot \frac{1}{1 + \frac{1}{2}\frac{f(x_k)}{f'(x_k)} f''(a_k)}$$

If x_k is close to ξ then $\left|\frac{1}{2}\frac{f(x_k)}{f'(x_k)} f''(a_k)\right| < 1$ so the denominator is non-zero.

Let $y_k = \frac{1}{2}\frac{f(x_k)}{f'(x_k)} f''(a_k)$. Then $\frac{1}{1+y} = 1 - y + \frac{y^2}{1+y}$ implies

$$e_{k+1} = e_k - \frac{f(x_k)}{f'(x_k)} \cdot \left(1 - y_k + \frac{y_k^2}{1+y_k}\right)$$

By Taylor's Theorem

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{1}{2}(\xi - x_k)^2 f''(b_k)$$

for some b_k between ξ and x_k. Therefore

$$\frac{f(x_k)}{f'(x_k)} = e_k - \frac{1}{2} e_k^2 \frac{f''(b_k)}{f'(x_k)} = e_k - e_k^2 A_k$$

where $A_k = \frac{f''(b_k)}{2f'(x_k)}$.

A first order approximation further yields

$$\frac{f(x_k)}{f'(x_k)} = e_k \frac{f'(c_k)}{f'(x_k)}$$

for some c_k between ξ and x_k. Thus

$$y_k = \frac{1}{2}\frac{f(x_k)}{f'(x_k)} f''(a_k) = e_k \frac{f'(c_k) f''(a_k)}{2f'(x_k)} = e_k B_k$$

where $B_k = \dfrac{f'(c_k) f''(a_k)}{2 f'(x_k)}$

Substituting yields

$$e_{k+1} = e_k - \left(e_k - e_k^2 A_k\right)\left(1 - e_k B_k + \frac{e_k^2 B_k^2}{1 + e_k B_k}\right)$$

$$= e_k - \left[e_k - e_k^2 B_k + e_k^3 \frac{B_k^2}{1+e_k B_k} - e_k^2 A_k + e_k^3 A_k B_k - e_k^4 \frac{A_k B_k^2}{1 + e_k B_k}\right]$$

$$= e_k^2 \left(B_k + A_k - e_k\left(\frac{B_k^2}{1+e_k B_k} + A_k B_k\right) + e_k^2 \frac{A_k B_k^2}{1+e_k B_k}\right)$$

Consequently $e_k \to 0$ as $k \to \infty$ yields

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty} \frac{f''(b_k)}{2f'(x_k)} = \frac{f''(\xi)}{2f'(\xi)}$$

$$\lim_{k\to\infty} B_k = \lim_{k\to\infty} \frac{f'(c_k) f''(a_k)}{2 f'(x_k)} = \frac{f'(\xi)\, f''(\xi)}{2 f'(\xi)} = \frac{f''(\xi)}{2}$$

and

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \lim_{k\to\infty}\left(B_k + A_k - e_k\left(\frac{B_k^2}{1+e_k B_k} + A_k B_k\right) + e_k^2 \frac{A_k B_k^2}{1+e_k B_k}\right)$$

$$= \frac{f''(\xi)}{2} + \frac{f''(\xi)}{2f'(\xi)} = \frac{1}{2} f''(\xi)\left(1 + \frac{1}{f'(\xi)}\right)$$

implies the convergence is quadratic.

Following the same template as in lab 2 the code

```
setprecision(4096)
f(x)=exp(x)-x-2
x0=big"1.0"

g(x)=x-f(x)^2/(f(x+f(x))-f(x))

xn=x0
xn1=g(xn)
en=abs(xn-xn1)
for n=1:9
    global xn,xn1,en
    xn=xn1
    xn1=g(xn)
    en0=en
    en=abs(xn-xn1)
    println("n=$(n-1) log(e(n+1))/log(e(n))=",
    Float64(log(en)/log(en0)))
end
```

with output

```
julia> include("c1p4.jl")
n=0 log(e(n+1))/log(e(n))=1.870977089541674
n=1 log(e(n+1))/log(e(n))=1.6527879299001056
n=2 log(e(n+1))/log(e(n))=1.8237197962494156
n=3 log(e(n+1))/log(e(n))=1.9062297930804768
n=4 log(e(n+1))/log(e(n))=1.950835847765074
n=5 log(e(n+1))/log(e(n))=1.9747984209782747
n=6 log(e(n+1))/log(e(n))=1.987238404308
n=7 log(e(n+1))/log(e(n))=1.9935782261130144
n=8 log(e(n+1))/log(e(n))=1.996778770051323
```

verifies that the convergence is quadratic.

1.6 Suppose that $f(\xi) = f'(\xi) = 0$, so that f has a double root at ξ, and that f'' is defined and continuous in a neighbourhood of ξ. If (x_k) is a sequence obtained by Newton's method, show that

$$\xi - x_{k+1} = -\tfrac{1}{2}\frac{(\xi - x_k)^2 f''(\eta_k)}{f'(x_k)} = \tfrac{1}{2}(\xi - x_k)\frac{f''(\eta_k)}{f''(\chi_k)},$$

where η_k and χ_k both lie between ξ and x_k. Suppose, further, that $0 < m < |f''(x)| < M$ for all x in the interval $[\xi - \delta, \xi + \delta]$ for some $\delta > 0$, where $M < 2m$; show that if x_0 lies in this interval the iteration converges to ξ, and that convergence is linear, with rate $\log_{10} 2$. Verify this conclusion by finding the solution of $e^x = 1 + x$, beginning from $x_0 = 1$.

The first equality comes the Taylor expansion

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{1}{2}(\xi - x_k)^2 f''(\eta_k)$$

for some η_k between ξ and x_k and is the same as equation (1.25) in the text.

Now apply Taylor to f' as

$$0 = f'(\xi) = f'(x_k) + (\xi - x_k) f''(\chi_k)$$

for some χ_k between ξ and x_k.

It follows that $f'(x_k) = -(\xi - x_k) f''(\chi_k)$. Substituting then yields

$$\xi - x_{k+1} = \frac{1}{2}(\xi - x_k)\frac{f''(\eta_k)}{f''(\chi_k)}$$

where η_k and χ_k both lie between ξ and x_k.

Suppose for some $\delta > 0$ that $x \in [\xi - \delta, \xi + \delta]$ implies

$$0 < m < |f''(x)| < M < 2m,$$

Claim that $x_0 \in [\xi - \delta, \xi + \delta]$ implies $x_k \to \xi$ as $k \to \infty$.

Let $e_k = x_k - \xi$. Then from the previous part $x_k \in [\xi - \delta, \xi + \delta]$ implies

$$|e_{k+1}| \leq \frac{1}{2}|e_k| \frac{|f''(\eta_k)|}{|f''(\chi_k)|} < \frac{1}{2}|e_k| \frac{M}{m} < \frac{1}{2}|e_k| \frac{2m}{m} = |e_k|$$

Thus $x_{k+1} \in [\xi - \delta, \xi + \delta]$ and by induction

$$|e_k| \leq \left(\frac{M}{2m}\right)^k |e_0| \to 0 \quad \text{as} \quad k \to \infty \quad \text{since} \quad \frac{M}{2m} < 1.$$

To find the rate note that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|} = \frac{1}{2} \lim_{k \to \infty} \frac{|f''(\eta_k)|}{|f''(\chi_k)|} = \frac{1}{2} = 10^{-p}$$

implies $p = \log_{10} 2$ is the asymptotic rate of convergence as given in Definition 1.4 from the text.

To verify the rate of convergence, note also that

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k} = \frac{1}{2} \lim_{k \to \infty} \frac{f''(\eta_k)}{f''(\chi_k)} = \frac{1}{2}$$

Therefore $e_{k+1} \sim \frac{1}{2} e_k$ as $k \to \infty$. Now

$$x_{k+2} - x_{k+1} = x_{k+2} - \xi + \xi - x_{k+1} = e_{k+2} - e_{k+1}$$

$$\approx \frac{1}{2} e_{k+1} - \frac{1}{2} e_k = \frac{1}{2}(x_{k+1} - x_k)$$

Thus,

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|} = \lim_{k \to \infty} \left| \frac{x_{k+2} - x_{k+1}}{x_{k+1} - x_k} \right| = \frac{1}{2}.$$

This may be verified by finding the roots of $f(x) = e^x - 1 - x$ starting with $x_0 = 1$.

The code

```
setprecision(4096)
f(x)=exp(x)-1-x
df(x)=exp(x)-1
x0=big"1.0"

g(x)=x-f(x)/df(x)

xn=x0
xn1=g(xn)
en=abs(xn-xn1)
for n=1:9
    global xn,xn1,en
    xn=xn1
    xn1=g(xn)
    en0=en
    en=abs(xn-xn1)
    println("n=$(n-1) e(n+1)/e(n)=",
    Float64(en/en0))
end
```

produces the output

```
julia> include("c1p6.jl")
n=0 e(n+1)/e(n)=0.6289641517089312
n=1 e(n+1)/e(n)=0.5745394449496848
n=2 e(n+1)/e(n)=0.5404998740256082
n=3 e(n+1)/e(n)=0.5211828256823164
n=4 e(n+1)/e(n)=0.5108436873236355
n=5 e(n+1)/e(n)=0.5054875624980186
n=6 e(n+1)/e(n)=0.5027605613806345
n=7 e(n+1)/e(n)=0.5013845208081329
n=8 e(n+1)/e(n)=0.500693326152156
```

which numerically verifies the limit is $1/2$.