HW3 problems 2.9, 2.10 and 2.12 due Nov 22

2.9 Prove that, for any nonsingular matrix $A \in \mathbb{R}^{n\times n}$,

$$\kappa_2(A) = \left(\frac{\lambda_n}{\lambda_1}\right)^{1/2},$$

where λ_1 is the smallest and λ_n is the largest eigenvalue of the matrix $A^{\mathrm{T}}A$.

Show that the condition number $\kappa_2(Q)$ of an orthogonal matrix Q is equal to 1. Conversely, if $\kappa_2(A) = 1$ for the matrix A, show that all the eigenvalues of $A^{\mathrm{T}}A$ are equal; deduce that A is a scalar multiple of an orthogonal matrix.

By definition $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$. Now by

Theorem 2.9 *Let $A \in \mathbb{R}^{n\times n}$ and denote the eigenvalues of the matrix $B = A^{\mathrm{T}}A$ by λ_i, $i = 1, 2, \ldots, n$. Then,*

$$\|A\|_2 = \max_{i=1}^{n} \lambda_i^{1/2}.$$

we have that

$$\|A\|_2 = \max\{\|Ax\|_2 : \|x\|_2 = 1\}$$

$$= \max\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } A^TA\} = \lambda_n^{1/2}$$

and similarly that

$$\|A^{-1}\|_2 = \max\{\|A^{-1}x\|_2 : \|x\|_2 = 1\}$$

$$= \max\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } (A^{-1})^TA^{-1}\}.$$

Since $(A^{-1})^T = (A^T)^{-1}$ then $(A^{-1})^TA^{-1} = (A^T)^{-1}A^{-1} = (AA^T)^{-1} = C^{-1}$ where $C = AA^T$. Suppose λ and x are an eigenvalue-eigenvector pair such that $Cx = \lambda x$. Then multiplying by C^{-1} yields that $x = C^{-1}\lambda x = \lambda C^{-1}x$ or equivalently

$$C^{-1}x = \frac{1}{\lambda}x.$$

Therefore the eigenvalues of C^{-1} are given by $\frac{1}{\lambda}$ where λ is an eigenvalue of C. It follows that

$$\|A^{-1}\|_2 = \max\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } (AA^T)^{-1}\}$$

$$= 1/\min\{\lambda^{1/2} ; \lambda \text{ is an eigenvalue of } AA^T\}.$$

To finish we need show that $C = AA^T$ and $B = A^TA$ have the same eigenvalues.

Suppose $Bx = \lambda x$. Define $y = Ax$. Then

$$Cy = CAx = AA^TAx = ABx = A\lambda x = \lambda Ax = \lambda y$$

shows that λ is an eigenvalue of C.

Similarly if $Cy = \lambda y$ then defining $x = A^Ty$ yields

$$Bx = BA^Ty = A^TAA^Ty = A^TCy = A^T\lambda y = \lambda A^Ty = \lambda x$$

and so λ is an eigenvalue of B. It follows that

$$\|A^{-1}\|_2 = 1/\min\{\lambda^{1/2} ; \lambda \text{ is an eigenvalue of } A^TA\} = \frac{1}{\lambda_1^{1/2}}$$

Consequently $\kappa_2(A) = \dfrac{\lambda_n^{1/2}}{\lambda_1^{1/2}} = \left(\dfrac{\lambda_n}{\lambda_1}\right)^{1/2}$ as desired.

2.10 Let $A \in \mathbb{R}^{n\times n}$. Show that if λ is an eigenvalue of $A^{\mathrm{T}}A$, then

$$0 \le \lambda \le \|A^{\mathrm{T}}\|\,\|A\|,$$

provided that the same subordinate matrix norm is used for both A and A^{T}. Hence show that, for any nonsingular $n \times n$ matrix A,

$$\kappa_2(A) \le \{\kappa_1(A)\,\kappa_\infty(A)\}^{1/2}.$$

Let λ and ξ be an eigenvalue-eigenvector pair such that

$$A^T A \xi = \lambda \xi.$$

Since $A^TA\xi \cdot \xi = A\xi \cdot A\xi = \|A\xi\|_2^2$ and $\lambda \xi \cdot \xi = \lambda \|\xi\|_2^2$ it follows that

$$\lambda = \frac{\|A\xi\|_2^2}{\|\xi\|_2} \geq 0$$

Now for any vector norm $\|\cdot\|$ and subordinate matrix norm $\|\cdot\|$ holds

$$\lambda \|\xi\| = |\lambda| \|\xi\| = \|\lambda \xi\| = \|A^T A \xi\| \le \|A^T\| \|A\| \|\xi\|.$$

Thus $\lambda \le \|A^T\| \|A\|$ and so $0 \le \lambda \le \|A^T\| \|A\|$.

We know that $\|A\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|$ and $\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$.

It follows that

$$\|A^T\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |[A^T]_{ij}| = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ji}| = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| = \|A\|_\infty$$

and

$$\|A^T\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |[A^T]_{ij}| = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ji}| = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}| = \|A\|_1$$

In particular $\|A^T\|_1 = \|A\|_\infty$ and $\|A^T\|_\infty = \|A\|_1$.

Now

$$\|A\|_2 = \max\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } A^TA\} \leq \|A^T\|_\infty^{1/2} \|A\|_\infty^{1/2} = \|A\|_1^{1/2} \|A\|_\infty^{1/2}$$

Similarly

$$\|A^{-1}\|_2 = \max\{\lambda^{1/2} : \lambda \text{ is an eigenvalue of } (A^{-1})^T(A^{-1})\} \leq \|(A^{-1})^T\|_\infty^{1/2} \|A^{-1}\|_\infty^{1/2}$$

$$= \|A^{-1}\|_1^{1/2} \|A^{-1}\|_\infty^{1/2} .$$

It follows that

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2 \leq \|A\|_1^{1/2} \|A\|_\infty^{1/2} \|A^{-1}\|_1^{1/2} \|A^{-1}\|_\infty^{1/2}$$

$$= \|A\|_1^{1/2} \|A^{-1}\|_1^{1/2} \|A\|_\infty^{1/2} \|A^{-1}\|_\infty^{1/2} = K_1(A)^{1/2} K_\infty(A)^{1/2}.$$

2.12 Let $B \in \mathbb{R}^{n\times n}$ and denote by I the identity matrix of order n. Show that if the matrix $I - B$ is singular, then there exists a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $(I - B)\boldsymbol{x} = \mathbf{0}$; deduce that $\|B\| \geq 1$, and hence that, if $\|A\| < 1$, then the matrix $I - A$ is nonsingular.

Now suppose that $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$. Show that

$$(I - A)^{-1} = I + A(I - A)^{-1} ,$$

and hence that

$$\|(I - A)^{-1}\| \leq 1 + \|A\| \, \|(I - A)^{-1}\| .$$

Deduce that

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|} .$$

If $I-B$ is singular then the nullspace of $I-B$ is non-trivial and so there is $x \neq 0$ such that $(I-B)x = 0$. Then $x = Bx$ and so

$$\|x\| = \|Bx\| \leq \|B\|\,\|x\|$$

implies after cancelling by $\|x\|$ that $\|B\| \geq 1$. Consequently, if $\|A\| < 1$ then $I-A$ is non-singular.

Now, multiplying

$$(I-A)\left(I + A(I-A)^{-1}\right) = I - A + (I-A)A(I-A)^{-1}$$

$$= I - A + (A - A^2)(I-A)^{-1} = I - A + A(I-A)(I-A)^{-1}$$

$$= I - A + AI = I - A + A = I$$

shows that $I + A(I-A)^{-1}$ is an inverse of $(I-A)$. Since inverses are unique, it follows that $(I-A)^{-1} = I + A(I-A)^{-1}$. Thus

$$\|(I-A)^{-1}\| \leq \|I\| + \|A(I-A)^{-1}\| \leq 1 + \|A\|\,\|(I-A)^{-1}\|$$

implies

$$(1 - \|A\|)\,\|(I-A)^{-1}\| \leq 1.$$

Therefore

$$\|(I-A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$