1. Let the function values $f(0)$, $f(1)$, $f(2)$ and $f(3)$ be given. We want to estimate

$$f(0.5), \qquad f'(1) \qquad \text{and} \qquad \int_0^3 f(x)dx.$$

(i) Let p be the cubic polynomial that interpolates these function values, and then approximate by

$$p(0.5), \qquad p'(1) \qquad \text{and} \qquad \int_0^3 p(x)dx.$$

Setting $x_i = i$, the Lagrange interpolating polynomial basis is given by

$$l_i(x) = \prod_{j \neq i} \frac{x-j}{i-j}$$

Specifically we have

$$l_0(x) = \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} = \frac{(x^2-3x+2)(x-3)}{-6}$$

$$= \frac{x^3-6x^2+11x-6}{-6} = -\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1$$

$$l_1(x) = \frac{x(x-2)(x-3)}{(1)(-1)(-2)} = \frac{x^3-5x^2+6x}{2} = \frac{1}{2}x^3 - \frac{5}{2}x^2 + 3x$$

$$l_2(x) = \frac{x(x-1)(x-3)}{(2)(1)(-1)} = \frac{x^3-4x^2+3x}{-2} = -\frac{1}{2}x^3 + 2x^2 - \frac{3}{2}x$$

$$l_3(x) = \frac{x(x-1)(x-2)}{(3)(2)(1)} = \frac{x^3-3x^2+2x}{6} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x$$

With this we can write

$$p(x) = \sum_{i=0}^{3} f(i)\, l_i(x) = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)x^3$$

$$+ \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)x^2 + \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)x + f(0)$$

$$p(0.5) = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)\frac{1}{8} + \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)\frac{1}{4}$$
$$+ \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)\frac{1}{2} + f(0)$$

$$= \left(\frac{-1}{48} + \frac{1}{4} - \frac{11}{12} + 1\right)f(0) + \left(\frac{1}{16} - \frac{5}{8} + \frac{3}{2}\right)f(1) + \left(-\frac{1}{16} + \frac{1}{2} - \frac{3}{4}\right)f(2) + \left(\frac{1}{48} - \frac{1}{8} + \frac{1}{6}\right)f(3)$$

$$= \frac{5}{16}f(0) + \frac{15}{16}f(1) - \frac{5}{16}f(2) + \frac{1}{16}f(3)$$

Therefore $f(0.5) \approx \frac{5}{16}f(0) + \frac{15}{16}f(1) - \frac{5}{16}f(2) + \frac{1}{16}f(3)$

$$p'(x) = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)3x^2 + \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)2x$$
$$+ \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)$$

$$p'(1) = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)3 + \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)2$$
$$+ \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)$$

$$\approx \left(-\frac{1}{2} + 2 - \frac{11}{6}\right)f(0) + \left(\frac{3}{2} - 5 + 3\right)f(1) + \left(-\frac{3}{2} + 4 - \frac{3}{2}\right)f(2) + \left(\frac{1}{2} - 1 + \frac{1}{3}\right)f(3)$$

$$\approx -\frac{1}{3}f(0) - \frac{1}{2}f(1) + f(2) - \frac{1}{6}f(3)$$

Therefore $f'(1) \approx -\frac{1}{3}f(0) - \frac{1}{2}f(1) + f(2) - \frac{1}{6}f(3)$

$$\int p(x)\,dx = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)\frac{x^4}{4} + \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)\frac{x^3}{3}$$
$$+ \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)\frac{x^2}{2} + f(0)x$$

$$\int_0^3 p(x)\,dx = \left(-\frac{1}{6}f(0) + \frac{1}{2}f(1) - \frac{1}{2}f(2) + \frac{1}{6}f(3)\right)\frac{81}{4} + \left(f(0) - \frac{5}{2}f(1) + 2f(2) - \frac{1}{2}f(3)\right)9$$
$$+ \left(-\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)\right)\frac{9}{2} + f(0)3$$

$$= \left(\frac{-81}{24}+9-\frac{99}{12}+3\right)f(0)+\left(\frac{81}{8}-\frac{45}{2}+\frac{27}{2}\right)f(1)+\left(-\frac{81}{8}+18-\frac{27}{4}\right)f(2)+\left(\frac{81}{24}-\frac{9}{2}+\frac{9}{6}\right)f(3)$$

$$\approx \frac{3}{8}f(0)+\frac{9}{8}f(1)+\frac{9}{8}f(2)+\frac{3}{8}f(3)$$

Therefore $\int_0^3 f(x)\,dx \approx \frac{3}{8}f(0)+\frac{9}{8}f(1)+\frac{9}{8}f(2)+\frac{3}{8}f(3)$.

(ii) Using the Lagrange formula, show that every approximation is a linear combination of the function values with constant coefficients and calculate these coefficients.

This was done already above. The results

$$f(0.5) \approx \frac{5}{16}f(0)+\frac{15}{16}f(1)-\frac{5}{16}f(2)+\frac{1}{16}f(3)$$

$$f'(1) \approx -\frac{1}{3}f(0)-\frac{1}{2}f(1)+f(2)-\frac{1}{6}f(3)$$

$$\int_0^3 f(x)\,dx \approx \frac{3}{8}f(0)+\frac{9}{8}f(1)+\frac{9}{8}f(2)+\frac{3}{8}f(3).$$

Note that the coefficients are constants.

(iii) Show the approximations are exact if f is any cubic polynomial.

Suppose $f(x) = ax^3+bx^2+cx+d$.

Then $f(0) = d$

$f(1) = a+b+c+d$

$f(2) = 8a+4b+2c+d$

$f(3) = 27a+9b+3c+d$

Now $f(0.5) = \frac{a}{8}+\frac{b}{4}+\frac{c}{2}+d$

while $\frac{5}{16}f(0) + \frac{15}{16}f(1) - \frac{5}{16}f(2) + \frac{1}{16}f(3)$

$= \frac{5}{16}d + \frac{15}{16}(a+b+c+d) - \frac{5}{16}(8a+4b+2c+d) + \frac{1}{16}(27a+9b+3c+d)$

$= \left(\frac{15}{16} - \frac{5}{2} + \frac{27}{16}\right)a + \left(\frac{15}{16} - \frac{5}{4} + \frac{9}{16}\right)b + \left(\frac{15}{16} - \frac{5}{8} + \frac{3}{16}\right)c + \left(\frac{5}{16} + \frac{15}{16} - \frac{5}{16} + \frac{1}{16}\right)d$

$= \frac{1}{8}a + \frac{1}{4}b + \frac{1}{2}c + d$

Since both answers are the same, then the formula for approximating $f(0.5)$ is exact when f is a cubic polynomial.

Next $f'(x) = 3ax^2 + 2bx + c$

so $f'(1) = 3a + 2b + c$

while $-\frac{1}{3}f(0) - \frac{1}{2}f(1) + f(2) - \frac{1}{6}f(3)$

$= -\frac{1}{3}d - \frac{1}{2}(a+b+c+d) + 8a+4b+2c+d - \frac{1}{6}(27a+9b+3c+d)$

$= \left(-\frac{1}{2} + 8 - \frac{9}{2}\right)a + \left(-\frac{1}{2} + 4 - \frac{3}{2}\right)b + \left(-\frac{1}{2} + 2 - \frac{1}{2}\right)c + \left(\frac{-1}{3} - \frac{1}{2} + 1 - \frac{1}{6}\right)d$

$= 3a + 2b + c$

Since both answers are the same, then the formula for approximating $f'(1)$ is exact when f is a cubic polynomial.

Next $\int (ax^3+bx^2+cx+d)\,dx = a\frac{x^4}{4} + b\frac{x^3}{3} + c\frac{x^2}{2} + dx$

so $\int_0^3 (ax^3+bx^2+cx+d)\,dx = \frac{81}{4}a + 9b + \frac{9}{2}c + 3d.$

While $\frac{3}{8}f(0)+\frac{9}{8}f(1)+\frac{9}{8}f(2)+\frac{3}{8}f(3)$

$=\frac{3}{8}d+\frac{9}{8}(a+b+c+d)+\frac{9}{8}(8a+4b+2c+d)+\frac{3}{8}(27a+9b+3c+d)$

$=\left(\frac{9}{8}+9+\frac{81}{8}\right)a+\left(\frac{9}{8}+\frac{9}{2}+\frac{27}{8}\right)b+\left(\frac{9}{8}+\frac{9}{4}+\frac{9}{8}\right)c+\left(\frac{3}{8}+\frac{9}{8}+\frac{9}{8}+\frac{3}{8}\right)d$

$=\frac{81}{4}a+9b+\frac{9}{2}c+3d$

Since both answers are the same, thus the formula for approximating $\int_0^3 f(x)\,dx$ is exact when f is a cubic polynomial.

(iv) Carry out this procedure for the function $f(x)=\sin(\pi x/6)$.

$\sin\left(\frac{\pi}{12}\right)\approx\frac{5}{16}\sin\left(\frac{\pi\cdot 0}{6}\right)+\frac{15}{16}\sin\left(\frac{\pi}{6}\right)-\frac{5}{16}\sin\left(\frac{\pi}{3}\right)+\frac{1}{16}\sin\left(\frac{\pi}{2}\right)$

$=0+\frac{15}{16}\cdot\frac{1}{2}-\frac{5}{16}\cdot\frac{\sqrt{3}}{2}+\frac{1}{16}\cdot 1=\frac{17-5\sqrt{3}}{32}\approx$

```
julia> (17-5*sqrt(3))/32
0.26061706131736295
```

The error is

$\frac{17-5\sqrt{3}}{32}-\sin\frac{\pi}{12}\approx$

```
julia> (17-5*sqrt(3))/32-sin(pi/12)
0.001798016214842213
```

$\frac{\pi}{6}\cos\left(\frac{\pi}{6}\right)\approx-\frac{1}{3}\sin\left(\frac{\pi\cdot 0}{6}\right)-\frac{1}{2}\sin\left(\frac{\pi}{6}\right)+1\sin\left(\frac{\pi}{3}\right)-\frac{1}{6}\sin\left(\frac{\pi}{2}\right)$

$=0-\frac{1}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}-\frac{1}{6}=\frac{-5+6\sqrt{3}}{12}\approx$

```
julia> (-5+6*sqrt(3))/12
0.449358737117772
```

The error is

$\frac{-5+6\sqrt{3}}{12}-\frac{\pi}{6}\cos\left(\frac{\pi}{6}\right)\approx$

```
julia> (-5+6*sqrt(3))/12-pi/6*cos(pi/6)
-0.004091103940782437
```

$$\int_0^3 \sin\frac{\pi x}{6}\,dx \approx \frac{3}{8}\sin\left(\frac{\pi\cdot 0}{6}\right)+\frac{9}{8}\cdot\sin\left(\frac{\pi}{6}\right)+\frac{9}{8}\sin\left(\frac{\pi}{3}\right)+\frac{3}{8}\sin\left(\frac{\pi}{2}\right)$$

$$= 0+\frac{9}{8}\cdot\frac{1}{2}+\frac{9}{8}\cdot\frac{\sqrt{3}}{2}+\frac{3}{8}\cdot 1 = \frac{15+9\sqrt{3}}{16} \approx$$

```
julia> (15+9*sqrt(3))/16
1.9117785792574935
```

Since

$$\int_0^3 \sin\frac{\pi x}{6}\,dx = -\frac{6}{\pi}\cos\frac{\pi x}{6}\Big|_0^3 = \frac{6}{\pi}$$

The error is

$$\frac{15+9\sqrt{3}}{16} - \frac{6}{\pi} \approx$$

```
julia> (15+9*sqrt(3))/16-6/pi
0.0019192621547494682
```

2. We shall approximate $f(x) = \sin(\pi x/6)$ by maping the Chebyshev nodes onto the interval $[0, 3]$ and then constructing the interpolating polynomial through these nodes.

(i) Find a $\ell(x) = ax + b$ such that $\ell\colon [-1, 1] \to [0, 3]$ is a bijection.

We need a line passing through $(-1,0)$ and $(1,3)$

$$l_1(x) = \frac{x-1}{-1-1} = -\frac{x}{2} + \frac{1}{2} \qquad l_2(x) = \frac{x+1}{1+1} = \frac{x}{2} + \frac{1}{2}$$

Now

$$l(x) = 0 \cdot l_1(x) + 3 \cdot l_2(x) = \frac{3}{2}x + \frac{3}{2}$$

Check $l(-1) = -\frac{3}{2} + \frac{3}{2} = 0$

$l(1) = \frac{3}{2} + \frac{3}{2} = \frac{6}{2} = 3$

(ii) Take $n = 3$ and find

$$x_k = \ell\left(\cos\left(\frac{\pi}{2}\frac{2k+1}{n+1}\right)\right) \qquad \text{for} \qquad k = 0, \ldots, n.$$

$$x_0 = l\left(\cos\left(\frac{\pi}{2}\frac{1}{4}\right)\right) = \frac{3}{2}\cos\frac{\pi}{8} + \frac{3}{2}$$

$$x_1 = \frac{3}{2}\cos\frac{3\pi}{8} + \frac{3}{2}$$

$$x_2 = \frac{3}{2}\cos\frac{5\pi}{8} + \frac{3}{2}$$

$$x_3 = \frac{3}{2}\cos\frac{7\pi}{8} + \frac{3}{2}$$

```
julia> x0=3/2*cos(pi/8)+3/2
2.88581929876693

julia> x1=3/2*cos(3*pi/8)+3/2
2.074025148547635

julia> x2=3/2*cos(5*pi/8)+3/2
0.9259748514523654

julia> x3=3/2*cos(7*pi/8)+3/2
0.11418070123307
```

(iii) Let q be the cubic polynomial that interpolates the function values $f(x_k)$ for $k = 0, \ldots, 3$, and then approximate by

$$q(0.5), \qquad q'(1) \qquad \text{and} \qquad \int_0^3 q(x)dx.$$

This was difficult to calculate by hand, so I used the Julia script. There are other ways to do this that are as good or better.

```
using Symbolics,SymbolicIntegration
@variables x,f0,f1,f2,f3
X=[3/2*cos(pi*(2*k+1)/8)+3/2 for k=0:3]
m(i)=prod(x-X[j+1] for j=0:3)/(x-X[i+1])
d(i)=substitute(m(i),x=>X[i+1])
l(i)=m(i)/d(i)
f=(f0,f1,f2,f3)
q=sum(f[i+1]*l(i) for i=0:3)
q05=substitute(q,x=>1//2)
fsin=(x->sin(pi*x/6)).(X)
fsubs=[f[i]=>fsin[i] for i=1:4]
approx=substitute(q05,fsubs)
E=approx-sin(pi/12)
```

← The polynomial
← evaluate polynomial at $x=0.5$
← substitute $\sin(\frac{\pi x}{6})$ in for f_i
← the approximation of $f(0.5)$
↑ The error

The output was.

```
julia> using Symbolics,SymbolicIntegration

julia> @variables x,f0,f1,f2,f3
5-element Vector{Num}:
  x
 f0
 f1
 f2
 f3

julia> X=[3/2*cos(pi*(2*k+1)/8)+3/2 for k=0:3]
4-element Vector{Float64}:
 2.88581929876693
 2.074025148547635
 0.9259748514523654
 0.11418070123307

julia> m(i)=prod(x-X[j+1] for j=0:3)/(x-X[i+1])
m (generic function with 1 method)

julia> d(i)=substitute(m(i),x=>X[i+1])
d (generic function with 1 method)

julia> l(i)=m(i)/d(i)
l (generic function with 1 method)

julia> f=(f0,f1,f2,f3)
(f0, f1, f2, f3)
```

```
julia> q=sum(f[i+1]*l(i) for i=0:3)
0.22677536732746076f0*(-0.9259748514523654 + x)*(-0.11418070123307 + x)*(-2.0740
25148547635 + x) - 0.5474841674140 96f1*(-0.9259748514523654 + x)*(-2.88581929876
693 + x)*(-0.11418070123307 + x) + 0.5474841674140959f2*(-2.88581929876693 + x)*
(-0.11418070123307 + x)*(-2.074025148547635 + x) - 0.22677536732746062f3*(-0.925
9748514523654 + x)*(-2.88581929876693 + x)*(-2.074025148547635 + x)

julia> q05=substitute(q,x=>1//2)
0.058664510801280 97f0 - 0.21467279915296816f1 + 0.7932402192848067f2 + 0.3627680
690668805f3

julia> fsin=(x->sin(pi*x/6)).(X)
4-element Vector{Float64}:
 0.9982134165725897
 0.8847498580883737
 0.4660661847984713
 0.059749267564360074

julia> fsubs=[f[i]=>fsin[i] for i=1:4]
4-element Vector{Pair{Num, Float64}}:
 f0 => 0.9982134165725897
 f1 => 0.8847498580883737
 f2 => 0.4660661847984713
 f3 => 0.059749267564360074

julia> approx=substitute(q05,fsubs)
0.2600055422257397

julia> E=approx-sin(pi/12)
0.0011864971232189436
```

for the derivative $f'(1)$ I added the additional lines

```
D(f,x)=expand_derivatives(Differential(x)(f))
dq=D(q,x)                                  ← the derivative g'(x)
dq1=substitute(dq,x=>1)
approx=substitute(dq1,fsubs)               ← approximation of f'(1)
E=approx-pi/6*cos(pi/6)                    ← error
```

```
julia> D(f,x)=expand_derivatives(Differential(x)(f))
D (generic function with 1 method)

julia> dq=D(q,x)
0.22677536732746076f0*(-0.11418070123307 + x)*(-2.074025148547635 + x) + 0.22677
536732746076f0*(-0.9259748514523654 + x)*(-0.11418070123307 + x) + 0.22677536732
746076f0*(-0.9259748514523654 + x)*(-2.074025148547635 + x) - 0.5474841674140 96f
1*(-0.9259748514523654 + x)*(-0.11418070123307 + x) - 0.5474841674140 96f1*(-0.92
59748514523654 + x)*(-2.88581929876693 + x) - 0.5474841674140 96f1*(-2.8858192987
6693 + x)*(-0.11418070123307 + x) + 0.5474841674140959f2*(-0.11418070123307 + x)
*(-2.074025148547635 + x) + 0.5474841674140959f2*(-2.88581929876693 + x)*(-0.114
18070123307 + x) + 0.5474841674140959f2*(-2.88581929876693 + x)*(-2.074025148547
635 + x) - 0.22677536732746062f3*(-0.9259748514523654 + x)*(-2.074025148547635 +
 x) - 0.22677536732746062f3*(-0.9259748514523654 + x)*(-2.88581929876693 + x) -
0.22677536732746062f3*(-2.88581929876693 + x)*(-2.074025148547635 + x)
```

```
julia> dq1=substitute(dq,x=>1)
-0.21891174322642004f0 + 0.955097231599766f1 - 0.3265578705450566f2 - 0.40962761
782828916f3

julia> approx=substitute(dq1,fsubs)
0.4498289700043638

julia> E=approx-pi/6*cos(pi/6)
-0.0036208710541906775
```

Approximating $\int_0^3 f(x)dx$ was similar

```
Iq=integrate(q,x)
Iq03=substitute(Iq,x=>3)-substitute(Iq,x=>0)
approx=substitute(Iq03,fsubs)
E=approx-6/pi
```

← the antiderivative

evaluate at the endpoints ↙

↖ approximation of $\int_0^3 f(x)dx$

Error ↙

```
julia> Iq=integrate(q,x)
0.22677536732746076f0*(-0.21928348052506014x + 1.1315186162670383(x^2) - 1.03806
02337443567(x^3) + (x^4) / 4) - 0.547484167414096f1*(-0.3051132241299661x + 1.55
37147127611153(x^2) - 1.3086582838174552(x^3) + (x^4) / 4) + 0.5474841674140959f
2*(-0.6834013893654373x + 3.2757901584040194(x^2) - 1.691341716182545(x^3) + (x^
4) / 4) - 0.22677536732746062f3*(-5.542201905979539x + 5.288976512567828(x^2) -
1.9619397662556437(x^3) + (x^4) / 4)

julia> Iq03=substitute(Iq,x=>3)-substitute(Iq,x=>0)
0.3964466094067268f0 + 1.103553390593276f1 + 1.1035533905932717f2 + 0.3964466094
06729f3

julia> approx=substitute(Iq03,fsubs)
1.9101233432005016

julia> E=approx-6/pi
0.0002640260977575526
```

(iv) Compare the approximations obtained from p in question 1(i) to the approximations given by q in 2(iii) above.

To compare the approximations, I computed the error in each one.

Quantity to approximate	Error in approximation obtained from p. (equally spaced)	Error in approximation obtained from q (Chebyshev)
$f(0.5)$	0.001798016214842213	0.0011864971232189436
$f'(1)$	-0.004091103940782437	-0.0036208710541906775
$\int_0^3 f(x)dx$	0.0019192621547494682	0.0002640260977575526

The error in the approximation of $f(0.5)$, $f'(1)$ and $\int_0^3 f(x)dx$ decreased when the Chebyshev nodes were used.

(v) Graph p, q and f on the same plot over the range $[0, 3]$.

```
using Plots
qsub=expand(substitute(q,fsubs))
qsin=eval(build_function(qsub,x))
m(i)=prod(x-j for j=0:3)/(x-i)
d(i)=substitute(m(i),x=>i)
l(i)=m(i)/d(i)
psub=expand(sum(sin(pi*i/6)*l(i) for i=0:3))
psin=eval(build_function(psub,x))
plot(x->sin(pi*x/6),0:0.01:3,label="f")
plot!(psin,0:0.01:3,label="p")
plot!(qsin,0:0.01:3,label="q")
```

convert the symbolic expression a compiled function.

redo problem 1 using Julia

convert to compiled function

Plot the polynomials.

```
julia> using Plots

julia> qsub=expand(substitute(q,fsubs))
-0.0011308810662045143 + 0.5354860519997273x - 0.018225345748676514(x^2) - 0.016
402130166003503(x^3)

julia> qsin=eval(build_function(qsub,x))
#11 (generic function with 1 method)

julia> m(i)=prod(x-j for j=0:3)/(x-i)
m (generic function with 1 method)

julia> d(i)=substitute(m(i),x=>i)
d (generic function with 1 method)

julia> l(i)=m(i)/d(i)
l (generic function with 1 method)

julia> psub=expand(sum(sin(pi*i/6)*l(i) for i=0:3))
0.534295227656675x - 0.017949192431122585(x^2) - 0.01634603522555267(x^3)

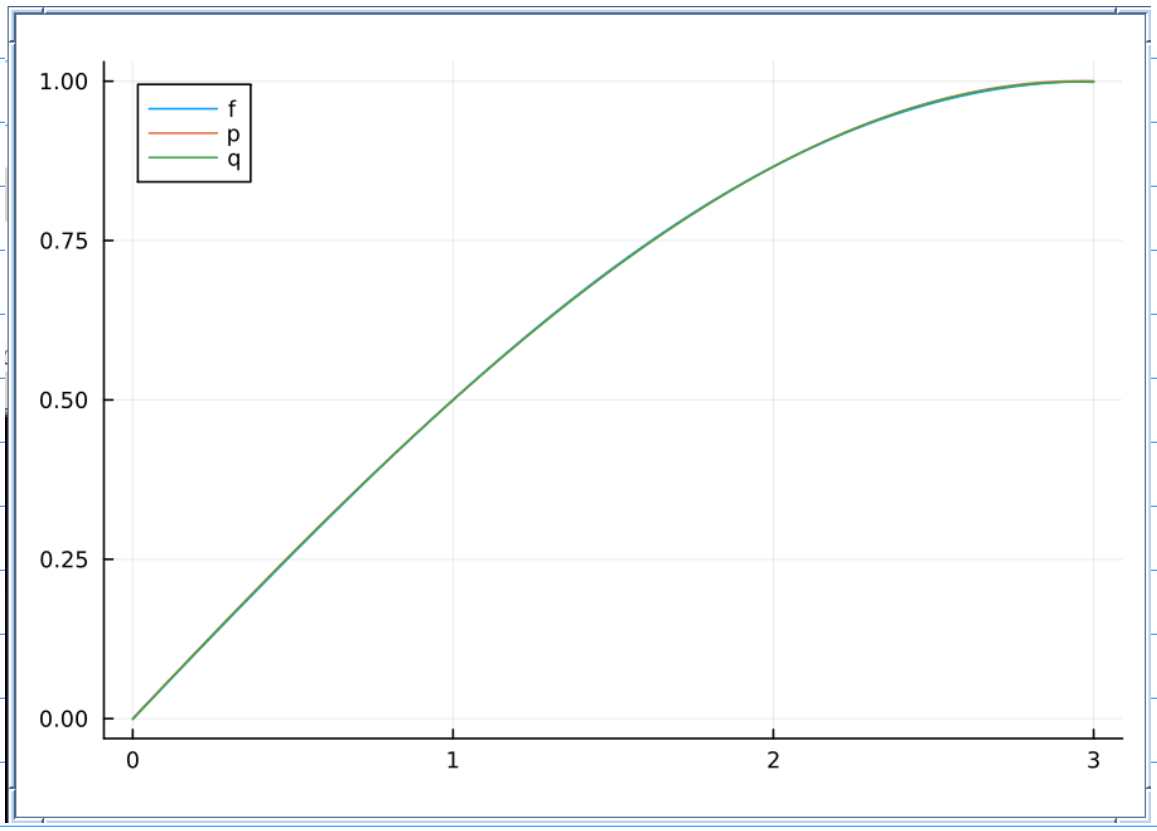
julia> psin=eval(build_function(psub,x))
#17 (generic function with 1 method)

julia> plot(x->sin(pi*x/6),0:0.01:3,label="f")

julia> plot!(psin,0:0.01:3,label="p")

julia> plot!(qsin,0:0.01:3,label="q")
```

The plots were as follows



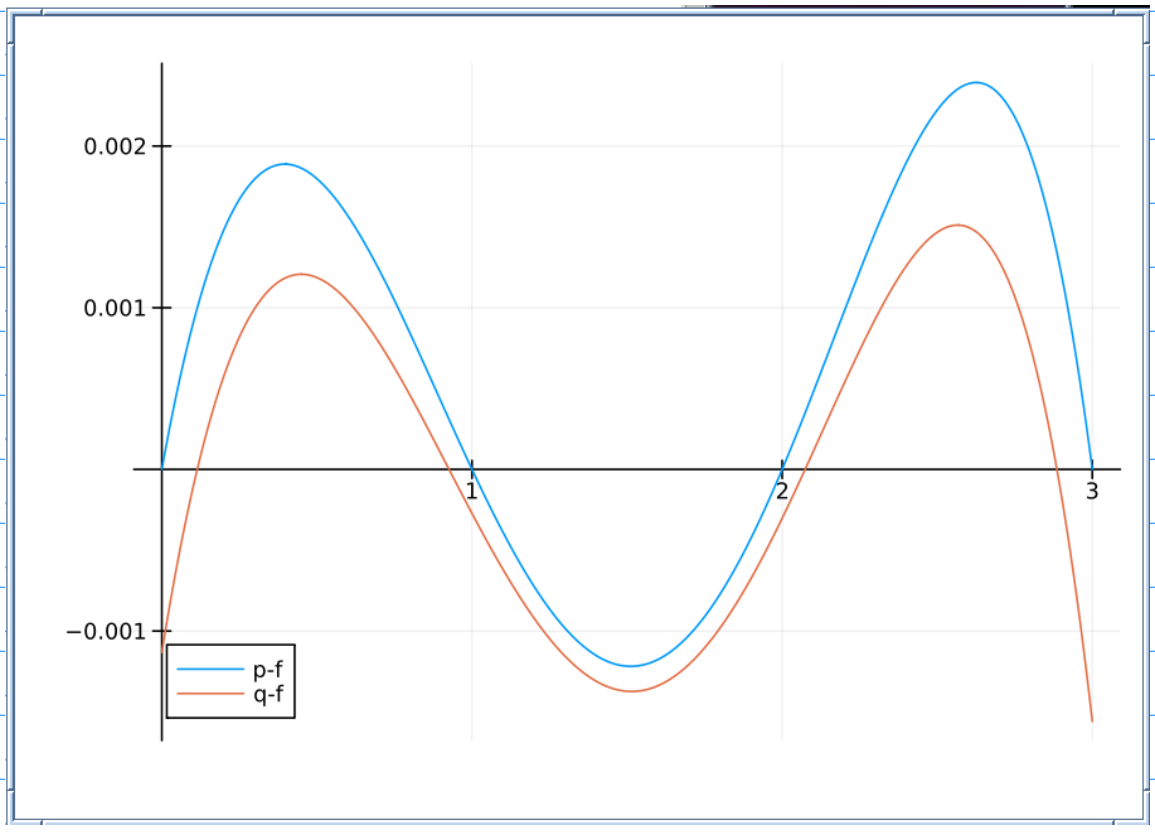
The graphs overlay each other in a visibly identical way. To see the differences more clearly, I also plotted the errors

$$E_p(t) = p(t) - f(t) \quad \text{and} \quad E_q(t) = q(t) - f(t)$$

```
plot(x->psin(x)-sin(pi*x/6),0:0.01:3,label="p-f")
plot!(x->qsin(x)-sin(pi*x/6),0:0.01:3,label="q-f",
   framestyle=:origin)
```

```
julia> plot(x->psin(x)-sin(pi*x/6),0:0.01:3,label="p-f")

julia> plot!(x->qsin(x)-sin(pi*x/6),0:0.01:3,label="q-f",
          framestyle=:origin)
```



The maximum error in E_q is smaller than the maximum error of E_p. Note that the errors are zero at the respective interpolation points which are different for p and q.

3. The functions $p_0, p_1, p_2 \ldots$ are generated by the Rodrigues formula

$$p_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}).$$

(i) Show that these functions are polynomials.

$p_0(x) = e^x x^0 e^{-x} = 1$

$p_1(x) = e^x \frac{d}{dx}(xe^{-x}) = e^x(e^{-x} - xe^{-x}) = 1 - x$

$p_2(x) = e^x \frac{d^2}{dx^2}(x^2 e^{-x}) = e^x \frac{d}{dx}(2xe^{-x} - x^2 e^{-x})$

$= e^x(2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x}) = 2 - 4x + x^2$

$p_3(x) = e^x \frac{d^3}{dx^3}(x^3 e^{-x}) = e^x \frac{d^2}{dx^2}(3x^2 e^{-x} - x^3 e^{-x})$

$= e^x \frac{d}{dx}(6xe^{-x} - 3x^2 e^{-x} - 3x^2 e^{-x} + x^3 e^{-x})$

$-6x^2 e^{-x}$

$= e^x \frac{d}{dx}(6e^{-x} - 6xe^{-x} - 12xe^{-x} + 6x^2 e^{-x} + 3x^2 e^{-x} - x^3 e^{-x})$

$= 6 - 18x + 9x^2 - x^3$

These are derivatives of products. Denoting the functions as f and g yields

$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

$\frac{d^2}{dx^2}(f(x)g(x)) = \frac{d}{dx}(f'(x)g(x) + f(x)g'(x)) = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)$

combine

$= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$

$$\frac{d^3}{dx^2}\left(f(x)g(x)\right) = \frac{d}{dx}\left(f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)\right)$$

$$= f'''(x)g(x) + f''(x)g'(x) + 2f''(x)g'(x) + 2f'(x)g''(x) + f'(x)g''(x) + f(x)g'''(x)$$

combine combine

$$= 1\cdot f'''(x)\,g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + 1\cdot f(x)g'''(x)$$

these are the binomial coefficients so following the pattern we have

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(x)\,g^{(k)}(x)$$

In Rodrigues formula we have $f(x) = x^n$ and $g(x) = e^{-x}$. It follows that

$$f^{(j)}(x) = \frac{d^j}{dx^j}x^n = n(n-1)\cdots(n-j+1)x^{n-j} = \frac{n!}{(n-j)!}x^{n-j}$$

thus

$$f^{(n-k)}(x) = \frac{d^{(n-k)}}{dx^{(n-k)}}x^n = \frac{n!}{k!}x^k$$

also

$$g^{(k)}(x) = \frac{d^k}{dx^k}e^{-x} = (-1)^k e^{-x}.$$

It follows that

$$\frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}x^k(-1)^k e^{-x}.$$

Consequently.

$$e^x \frac{d^n}{dx^n}(x^n e^{-x}) = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} x^k (-1)^k e^{-x}.$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (-1)^k x^k$$

which is a polynomial of degree n with coefficients $\binom{n}{k} \frac{n!}{k!}(-1)^k$.

(ii) Prove by integration by parts that for every polynomial p of degree less or equal $n-1$ we have the orthogonality condition $\langle p_n, p\rangle = 0$ with respect to the scalar product given by

$$\langle f, g\rangle = \int_0^\infty e^{-x} f(x) g(x) dx.$$

Thus, these polynomials are the Laguerre polynomials.

By definition

$$\langle p_n, p\rangle = \int_0^\infty e^{-x} p_n(x) p(x)\, dx = \int_0^\infty e^{-x} e^{x} \frac{d^n}{dx^n}(x^n e^{-x}) p(x)\, dx$$

$$= \int_0^\infty \frac{d^n}{dx^n}(x^n e^{-x}) p(x)\, dx.$$

Now integrate by parts.

$u = p(x)$ $\qquad du = p'(x)dx$

$dv = \frac{d^n}{dx^n}(x^n e^{-x})$ $\qquad v = \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})$

$$\langle p_n, p\rangle = p(x) \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\Big|_0^\infty - \int_0^\infty \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) p'(x)\, dx.$$

Note that $\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})$ has an e^{-x} in every term and since there are only $n-1$ derivative that there is also x in every term.

It follows that

$$\left.\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\right|_{x=0} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) = 0$$

Therefore

$$\langle p_n, p\rangle = -\int_0^\infty \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\, p'(x)\, dx.$$

This integration by parts procedure can be repeated $n-1$ more times to obtain

$$\langle p_n, p\rangle = (-1)^n \int_0^\infty x^n e^{-x}\, p^{(n)}(x)\, dx.$$

Since p is a polynomial of at most $n-1$ it follows that $p^{(n)}(x) = 0$. Consequently

$$\langle p_n, p\rangle = 0.$$

(iii) Calculate p_3, p_4 and p_5 from the Rodrigues formula.

Expanded out as before, Rodrigues formula is

$$p_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!}(-1)^k x^k$$

$$p_3(x) = \sum_{k=0}^{3} \binom{3}{k} \frac{3!}{k!}(-1)^k x^k$$

Binomial coefficients can be obtained from Pascal's triangle

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$\binom{3}{k}$ $\binom{4}{k}$ $\binom{5}{k}$

also

Table of Factorials

$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$

Therefore

$$p_3(x) = 6 + 3\frac{6}{1}(-1)x + 3\frac{6}{2}x^2 + 1\frac{6}{6}(-1)x^3$$

$$= 6 - 18x + 9x^2 - x^3$$

Note, this agrees with the value of $p_3(x)$ computed earlier.

$$p_4(x) = 24 - 4\frac{24}{1}x + 6\frac{24}{2}x^2 - 4\frac{24}{6}x^3 + 1\frac{24}{24}x^4$$

$$= 24 - 96x + 72x^2 - 16x^3 + x^4$$

$$p_5(x) = 120 - 5\frac{120}{1}x + 10\frac{120}{2}x^2 - 10\frac{120}{6}x^3 + 5\frac{120}{24}x^4 - 1\frac{120}{120}x^5$$

$$= 120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5$$