HW4 problems 4.1 and 4.2 due Dec 6

4.1 Suppose that the function $\boldsymbol{g}$ is a contraction in the ∞-norm, as in (4.5). Use the fact that

$$\|\boldsymbol{g}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{y})\|_p \le n^{1/p}\|\boldsymbol{g}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{y})\|_\infty$$

to show that $\boldsymbol{g}$ is a contraction in the p-norm if $L < n^{-1/p}$.

Let $z = g(x)-g(y)$. By definition since $|z_i| \le \|z\|_\infty$ then

$$\|z\|_p = \left(\sum_{i=1}^n |z_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n \|z\|_\infty^p\right)^{1/p} = \left(n\|z\|_\infty^p\right)^{1/p} = n^{1/p}\|z\|_\infty .$$

Consequently

$$\|g(x)-g(y)\|_p \le n^{1/p}\|g(x)-g(y)\|_\infty .$$

Now suppose g is a contraction in the ∞-norm

$$\|g(x)-g(y)\|_\infty \le L\|x-y\|_\infty$$

with $L < n^{-1/p} \le 1$. Then $\gamma = Ln^{1/p} < 1$ and

$$\|g(x)-g(y)\|_p \le n^{1/p}\|g(x)-g(y)\|_\infty \le n^{1/p} L\|x-y\|_\infty = \gamma\|x-y\|_\infty$$

Now $|z_i| \le \|z\|_p$ implies

$$\|z\|_\infty = \max\{|z_1|,|z_2|,\dots,|z_n|\} \le \|z\|_p$$

and so

$$\|g(x)-g(y)\|_p \le \gamma\|x-y\|_p$$

where $\gamma < 1$ shows g is a contraction in the p norm.

4.2 Show that the simultaneous equations $\boldsymbol{f}(x_1, x_2) = \mathbf{0}$, where $\boldsymbol{f} = (f_1, f_2)^{\mathrm{T}}$, with

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 25\,, \qquad f_2(x_1, x_2) = x_1 - 7x_2 - 25\,,$$

have two solutions, one of which is $x_1 = 4$, $x_2 = -3$, and find the other. Show that the function $\boldsymbol{f}$ does not satisfy the conditions of Theorem 4.3 at either of these solutions, but that if the sign of f_2 is changed the conditions are satisfied at one solution, and that if $\boldsymbol{f}$ is replaced by $\boldsymbol{f}^* = (f_2 - f_1, -f_2)^{\mathrm{T}}$, then the conditions are satisfied at the other. In each case, give a value of the relaxation parameter λ which will lead to convergence.

First solve

$$\begin{cases} x_1^2 + x_2^2 = 25 \\ x_1 - 7x_2 = 25 \end{cases}$$

Substitute $x_1 = 7x_2 + 25$ yields

$$(7x_2 + 25)^2 + x_2^2 = 25$$

$$49x_2^2 + 350x_2 + 625 + x_2^2 = 25$$

$$50x_2^2 + 350x_2 + 600 = 0$$

$$5(10x_2^2 + 70x_2 + 120) = 0$$

$$5^2(2x_2^2 + 14x_2 + 24) = 0$$

$$5^2(2x_2 + 8)(x_2 + 3) = 0$$

so $x_2 = -4$ or -3

$25 \times 7 = 175$, $175 \times 2 = 350$
$25 \times 25 = 125 + 500 = 625$
$600 / 5 = 120$
$350 / 5 = 70$
$70 / 5 = 14$

If $x_2 = -3$ then $x_1 = 7(-3)+25 = -21+25 = 4$ which is the solution $x_1 = 4$ and $x_2 = -3$ already provided.

If $x_2 = -4$ then $x_1 = 7(-4)+25 = -28+25 = -3$ and so the solution $x_1 = -3$ and $x_2 = -4$ is the other solution.

Recall

Theorem 4.3 *Suppose that* $\boldsymbol{f}(\boldsymbol{\xi}) = \mathbf{0}$, *and that all the first partial derivatives of* $\boldsymbol{f} = (f_1, \ldots, f_n)^{\mathrm{T}}$ *are defined and continuous in some (open) neighbourhood of* $\boldsymbol{\xi}$, *and satisfy a condition of strict diagonal dominance at* $\boldsymbol{\xi}$; *i.e.,*

$$\frac{\partial f_i}{\partial x_i}(\boldsymbol{\xi}) > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{\partial f_i}{\partial x_j}(\boldsymbol{\xi}) \right|, \quad i = 1, 2, \ldots, n\,. \tag{4.17}$$

Then, there exist $\varepsilon > 0$ *and a positive constant* λ *such that the relaxation iteration (4.16) converges to* $\boldsymbol{\xi}$ *for any* $\boldsymbol{x}_0$ *in the closed ball* $\bar{B}_\varepsilon(\boldsymbol{\xi})$ *of radius* ε, *centre* $\boldsymbol{\xi}$.

Since $f_1(x_1, x_2) = x_1^2 + x_2^2 - 25$ and $f_2(x_1, x_2) = x_1 - 7x_2 - 25$ then

$$Df(x) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ 1 & -7 \end{bmatrix}$$

When $\xi = (4,-3)$ then

$$Df(4,-3) = \begin{bmatrix} 8 & -6 \\ 1 & -7 \end{bmatrix}$$

and $8 > |-7| = 7$ implies (4.17) holds for $i=1$

but $-7 \not> |1|$ implies (4.17) does not hold for $i=2$

Therefore f doesn't satisfy the hypothesis of Theorem 4.3 at $\xi = (4,-3)$.

When $\xi = (-3,-4)$ then

$$Df(-3,-4) = \begin{bmatrix} 2x_1 & 2x_2 \\ 1 & -7 \end{bmatrix}\Bigg|_{(x_1,x_2)=(-3,-4)} = \begin{bmatrix} -6 & -8 \\ 1 & -7 \end{bmatrix}$$

and the situation is worse since $-6 \not> |-8| = 8$ so the hypothesis of Theorem 4.3 don't hold for either $i=1$ or $i=2$.

Changing the sign of f_2 and considering

$$f_1(x_1,x_2) = x_1^2 + x_2^2 - 25 \quad \text{and} \quad f_2(x_1,x_2) = -x_1 + 7x_2 + 25$$

yields

$$Df(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ -1 & 7 \end{bmatrix}$$

in which case

$$Df(4,-3) = \begin{bmatrix} 8 & -6 \\ -1 & 7 \end{bmatrix}$$

Now $8 > |-6|$ and $7 > |-1|$ shows $Df(4,-3)$ is diagonally dominant and so there is a λ such that the iteration $x^{(k+1)} = g(x^{(k)})$ where $g(x) = x - \lambda f(x)$ converges.

According to the proof of the theorem in this case we may take

$$\lambda = 1/m = \frac{1}{\max\limits_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(\xi)} = \frac{1}{\max(8,7)} = \frac{1}{8}.$$

Writing $f^*(x) = (f_2 - f_1, -f_2)$ yields

$$Df^*(x) = \begin{bmatrix} 1-2x_1 & -7-2x_2 \\ -1 & 7 \end{bmatrix}$$

and so

$$Df^*(-3,-4) = \begin{bmatrix} 1+6 & -7+8 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -1 & 7 \end{bmatrix}$$

since $7 > 1$ this is diagonally dominant and we may take

$$\lambda = \frac{1}{m} = \frac{1}{\max(7,7)} = \frac{1}{7}.$$