In 1669 Isaac Newton devised a technique for approximating the solution of a polynomial equation [2]. In 1685 John Wallis named this method Newton’s method and Joseph Raphson simplified it in 1690. In 1740 Thomas Simpson extended the method to general nonlinear equations and systems of equations [3]. In 2000 Dongarra and Sullivan listed Newton’s method among the top 10 algorithms of the 20th century [1].

**Newton’s Method**

Newton’s method is given by the fixed point iteration

\[ x_{n+1} = g(x_n) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)} \]

and \( x_0 \) is an initial approximation of the root.

**Convergence of Newton’s Method.** Let \( f \) be a twice continuously differentiable function. Let \( a \) be a point such that \( f(a) = 0 \) and \( f'(a) \neq 0 \). Prove that Newton’s method is quadratically convergent provided \( x_0 \) is close enough to \( a \).

**Proof.** Let \( \delta > 0 \) be chosen small enough such that

\[ |g'(x)| = \left| \frac{f(x)f''(x)}{f'(x)^2} \right| \leq \gamma < 1 \quad \text{for} \quad |x - a| \leq \delta. \]

Then, provided \( |x_0 - a| \leq \delta \), the inequality

\[ |x_{n+1} - a| = |g(x_n) - g(a)| = \left| \int_a^{x_n} g'(s) ds \right| \leq \gamma |x_n - a| \]

shows \( |x_n - a| \leq \gamma^n |x_0 - a| \to 0 \) as \( n \to \infty \) and moreover that \( |x_n - a| \leq \delta \). Now define \( e_n = x_n - a \). By Taylor’s theorem there exists \( \xi_n \) between \( x_n \) and \( a \) such that

\[ 0 = f(a) = f(x_n) - f'(x_n)e_n + \frac{f''(\xi_n)}{2} e_n^2 \quad \text{for} \quad n = 0, 1, 2, \ldots. \]
Therefore
\[ \frac{f(x_n)}{f'(x_n)} = e_n - \frac{f''(\xi_n)}{2f'(x_n)}e_n^2. \]

It follows that
\[ e_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - a = \frac{f''(\xi_n)}{2f'(x_n)}e_n^2. \]

Let
\[ A = \max \{|f''(x)| : |x - a| \leq \delta\} \quad \text{and} \quad B = \min \{|f'(x)| : |x - a| \leq \delta\}. \]

Since \( f'' \) is continuous then \( A < \infty \). By definition of \( \delta \) we have \( f'(x) \neq 0 \) for \( |x - a| \leq \delta \). Therefore, continuity of \( f' \) implies \( B > 0 \). It follows that
\[ |e_{n+1}| = \left| \frac{f''(\xi_n)}{2f'(x_n)}e_n^2 \right| \leq \frac{A}{2B}|e_n|^2 \quad \text{for} \quad n = 0, 1, 2, \ldots. \]

Consequently \( |e_{n+1}| \leq M|e_n|^2 \) where \( M = A/(2B) \). This shows Newton’s method is at least quadratically convergent.

It is sometimes said that Newton’s method doubles the number of significant digits at each iteration. This can be explained as follows: Let
\[ \alpha = \log_{10}(5M|a|) \quad \text{so that} \quad 10^\alpha = 5M|a|. \]

Suppose \( x_n \) is accurate to \( k \) significant digits. By the definition this means
\[ \frac{|x_n - a|}{|a|} \leq 5 \times 10^{-k}. \]

Now
\[ \frac{|x_{n+1} - a|}{|a|} \leq \frac{M|x_n - a|^2}{|a|} = M|a|\left(\frac{|x_n - a|}{|a|}\right)^2 \leq M|a|(5^2 \times 10^{-2k}) = 5 \times 10^{\alpha-2k} \]
implies \( x_{n+1} \) is accurate to \( 2k - \alpha \) significant digits. Provided \( k \) is large compared to \( \alpha \) this is about twice the number of significant digits that were accurate in \( x_n \). Since \( k \to \infty \) as \( x_n \to a \), it is natural to assume that \( k \) is very large compared to \( \alpha \). Therefore Newton’s method about doubles the number of significant digits between each iteration.

References
2. Isaac Newton, *De analysi per aequationes numero terminorum infinitas*, 1669.