A Summary of Results on Newton's Method.

Given an equation $f(x)=0$ an an intial approximation of the solution x_0, Newton's method uses the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to construct a sequence of approximations.

Theorem: Suppose f is two times continuously differentiable and that there is a solution α such that $f(\alpha)=0$ and for which $f'(\alpha)\neq 0$. If x_0 is sufficiently close to α then the sequence $x_n \to \alpha$ quadratically as $n\to\infty$.

On Friday September 29 at computing lab 3 we showed that if $x_n \to \alpha$ that convergence is quadratic. On Monday Oct 2 we showed x_n converges provided that x_0 is close enough to α. We now put those two results together to prove the theorem.

Let $g(x) = x - \frac{f(x)}{f'(x)}$. Then $x_{n+1} = g(x_n)$ and we know x_n converges if $|g'(x)|<1$ for $x \in (\alpha-\delta, \alpha+\delta)$ and $x_0 \in (\alpha-\delta,\alpha+\delta)$.

Differentiating yields

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

Therefore $|g'(x)| < 1$ follows when

$$|f(x) f''(x)| < |f'(x)|^2.$$

Since $|f(\alpha) f''(\alpha)| = 0 |f''(\alpha)| = 0$ and $|f'(\alpha)|^2 > 0$ then

$$|f(\alpha) f''(\alpha)| < |f'(\alpha)|^2.$$

Now since f, f' and f'' are continuous by assumption, there exists $\delta > 0$ such that

$$|f(x) f''(x)| < |f'(x)|^2 \quad \text{for} \quad x \in (\alpha-\delta, \alpha+\delta).$$

Consequently

$$|g'(x)| < 1 \qquad \text{for} \qquad x \in (\alpha-\delta, \alpha+\delta).$$

and it follows that $x_n \to \alpha$ provided $x_0 \in (\alpha-\delta, \alpha+\delta)$.

At this point we know x_n converges so the only thing left is to show that convergence is quadratic.

By Taylor's theorem

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(c_n)$$

where c_n is between α and x_n.

Defining $e_n = \alpha - x_n$ and recalling that $f(\alpha) = 0$ yields

$$0 = f(x_n) + e_n f'(x_n) + \frac{e_n^2}{2} f''(c_n)$$

Since $x_n \in (\alpha-\delta, \alpha+\delta)$ then $f'(x_n) \neq 0$ and dividing obtains

$$\frac{f(x_n)}{f'(x_n)} = -e_n - \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}$$

Now

$$e_{n+1} = \alpha - x_{n+1} = \alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) = e_n + \frac{f(x_n)}{f'(x_n)}$$

$$= \cancel{e_n} - \cancel{e_n} - \frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)} = -\frac{e_n^2}{2}\frac{f''(c_n)}{f'(x_n)}$$

Since $x_n \to \alpha$ and c_n is between α and x_n then $c_n \to \alpha$.

Therefore

$$e_{n+1} \approx -\frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} \quad \text{as } n \to \infty.$$

In other words, there is an bound M such that

$$|e_{n+1}| \leq M|e_n|^2 \quad \text{for all } n.$$

The exponent on $|e_n|$ being 2 means the convergence is quadratic.