Bernoulli Numbers Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. The first computer program ever written was by Ada Lovelace who wrote a program for the Analytical Engine to compute Bernoulli numbers. The Bernoulli number B_n is given by $B_n = \mathcal{B}_n(0)$ where $\mathcal{B}_n(x)$ is the unique polynomial of degree n such that

$$\int_x^{x+1} \mathcal{B}_n(t)dt = x^n.$$

Find B_n for $n = 0$, 1, 2 and 3 by substituting a polynomial of degree n into the integral and solving for the coefficients so that equality holds. You may use Maple or some other computer algebra system or do the calculation by hand.

b. By the Fundamental Theorem of Calculus it follows that

$$\frac{d}{dx}\int_x^{x+1} \mathcal{B}_n(t)dt = \mathcal{B}_n(x+1) - \mathcal{B}_n(x) = \int_x^{x+1} \mathcal{B}_n'(t)dt.$$

Use this fact to show that $\mathcal{B}_n'(x) = n\mathcal{B}_{n-1}(x)$.

c. By the Fundamental Theorem of Calculus we also have

$$\int_0^x \mathcal{B}_n'(t)dt = \mathcal{B}_n(x) - \mathcal{B}_n(0) \quad \text{or equivalently} \quad \mathcal{B}_n(x) = B_n + \int_0^x n\mathcal{B}_{n-1}(t)dt.$$

Integrate the above equality in x from 0 to 1, then interchange the order of integration to obtain the relation that

$$B_n = \int_0^1 tn\mathcal{B}_{n-1}(t)dt \qquad \text{for} \qquad n > 1.$$

d. Write $\mathcal{B}_{n-1}(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$ and use the identity

$$\mathcal{B}_n(x) = \int_0^1 tn\mathcal{B}_{n-1}(t)dt + \int_0^x n\mathcal{B}_{n-1}(t)dt$$

derived in the previous step to find formulas for B_n and $\mathcal{B}_n(x)$ in terms of the α_k.

e. Starting with $\mathcal{B}_1(x) = x - 1/2$, write a program that computes the Bernoulli numbers by means of the formulas derived in the previous step. Use your program to print a table listing the values of n and B_n for $n = 1, 2, \ldots, 10$.

f. Comment on the accumulation of rounding errors that result from using the formulas derived above to compute the Bernoulli numbers. Suggest a way to reduce rounding errors and test it numerically. What algorithm did Ada Lovelace use in the program she wrote for the Analytical Engine?