Approximation of Derivatives by FFT Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function with period 2. Approximate f on the interval $[-1, 1]$ as $f \approx A$ where

$$A(x) = \sum_{j=-N/2+1}^{N/2} y_j e^{i\pi jx} \quad \text{and} \quad y_j = \frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} f\Big(\frac{2\ell}{N}\Big) e^{-2\pi i \ell j/N}.$$

Differentiate A to obtain approximations for f', f'' and f'''.

b. Modify the approximation in part 1 by setting $y_{N/2} = 0$ to obtain $\widetilde{A}$. Explain why $\widetilde{A}$ is guaranteed to be real for all values of x.

c. Let $f(x) = \exp(\sin \pi x)$. Compute f', f'' and f''' exactly.

d. Define

$$A'_\ell = A'\Big(\frac{2\ell}{N}\Big), \qquad A''_\ell = A''\Big(\frac{2\ell}{N}\Big), \qquad A'''_\ell = A'''\Big(\frac{2\ell}{N}\Big),$$
$$\widetilde{A}'_\ell = \widetilde{A}'\Big(\frac{2\ell}{N}\Big), \qquad \widetilde{A}''_\ell = \widetilde{A}''\Big(\frac{2\ell}{N}\Big), \qquad \widetilde{A}'''_\ell = \widetilde{A}'''\Big(\frac{2\ell}{N}\Big).$$

Write a program that uses the the FFT and inverse FFT to compute these approximations for $N = 4, 8, 16$. Display your results in a table form. Are the imaginary parts of A'_ℓ, A''_ℓ and A'''_ℓ zero? How about the imaginary parts of $\widetilde{A}'_\ell$, $\widetilde{A}''_\ell$ and $\widetilde{A}'''_\ell$? Which are better approximations? What role does rounding error play?

e. Compute the errors

$$E_k = \left(\frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} \left| A_\ell^{(k)} - f^{(k)}\Big(\frac{2\ell}{N}\Big) \right|^2 \right)^{1/2}$$

and

$$\widetilde{E}_k = \left(\frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} \left| \widetilde{A}_\ell^{(k)} - f^{(k)}\Big(\frac{2\ell}{N}\Big) \right|^2 \right)^{1/2}$$

for $k = 1, 2, 3$ and $N = 4, 8, \ldots, 65536$. Comment on the quality of the approximations.

f. Repeat parts 3 through 5 above for the function $f(x) = \exp(x^2)$. Compare the size of the errors and rate of convergence as $N \to \infty$ in this case to the previous one. Explain any differences or similarities.