The Foias Sequence Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Given $\alpha \in [1, 2]$ consider the recurrence relation

$$x_{n+1} = \left(1 + \frac{1}{x_n}\right)^n \qquad \text{where} \qquad x_1 = \alpha.$$

Set $\alpha = 1$ and write a computer program in Julia that computes x_n for $n = 2, \ldots, 50$. Please include the full program listing and output. For reference,

$$x_{15} \approx 1.2634346914781789 \qquad \text{and} \qquad x_{16} \approx 6283.87526674291.$$

b. Based on the numerical evidence in the previous step, make a conjecture regarding the value of the limits

$$\lim_{n\to\infty} x_{2n} \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n+1} \qquad \text{when} \qquad x_1 = 1.$$

Explain your reasoning in as much mathematical detail as possible.

c. Change the program in the previous step to compute the values of x_n when $\alpha = 2$. Look at the output and now make a conjecture regarding the value of the limits

$$\lim_{n\to\infty} x_{2n} \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n+1} \qquad \text{when} \qquad x_1 = 2.$$

Explain the reasoning behind your conjecture.

d. Define

$$\alpha_* = \sup\left\{\alpha : |x_{2n+1}| \text{ is bounded as } n \to \infty \right\}.$$

Intuitively α_* is the largest value of α such that $|x_{2n+1}|$ is bounded. Explain in details how the interval bisection method could be used to approximate α_*.

e. Write a computer program that bisects the interval $[1, 2]$ to find an approximation of α_* good to at least 4 significant digits. Include the full program listing and output.

f. Explain theoretically how many times the interval $[1, 2]$ needs to be bisected to ensure the resulting approximation is good to at least 4 significant digits. What if 6 significant digits are desired? How about 8 significant digits?

g. Define

$$\alpha^* = \inf \Big\{ \alpha : |x_{2n}| \text{ is bounded as } n \to \infty \Big\}.$$

Intuitively α^* is the smallest value of α such that $|x_{2n}|$ is bounded. Modify the program from the previous step to obtain an approximation of α^*. Include the output and describe what modifications were made to the code.

h. Based on the numerical evidence obtained in the previous two problems conjecture whether the values of α_* and α^* are equal or different.