Halley's Method Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

Halley's method is the fixed-point iteration for solving $f(x) = 0$ given by $x_{n+1} = h(x_n)$ where

$$h(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}.$$

1a. Show that $f(x) = 0$ implies $h(x) = x$. Conversely show that if $h(x) = x$ then either $f(x) = 0$ or $f'(x) = 0$.

b. Compute $h'(x)$ and show that

$$h'(x) = \begin{cases} 0 & \text{when } f(x) = 0 \text{ and } f'(x) \neq 0 \\ 2 & \text{when } f(x) \neq 0,\ f'(x) = 0 \text{ and } f''(x) \neq 0. \end{cases}$$

Conclude that the fixed points of h for which $f(x) = 0$ are stable, but the fixed points for which $f'(x) = 0$ are not.

c. Use this fixed point iteration with $x_0 = 1$ to solve $f(x) = 0$ where $f(x) = x^2 - 2$. Compare the performance of this method with Newton's method. Which is faster?

d. The function

$$f(x) = 2\cos(5x) + 2\cos(4x) + 6\cos(3x) + 4\cos(2x) + 10\cos(x) + 3.$$

has two roots on $[0, 3]$; one root is near 1 and the other near 2. Compare Halley to Newton when finding the roots of this equation.

e. Suppose $f(x) = (x - \xi)^m q(x)$ where $\lim_{x\to\xi} q(x) \neq 0$. Let $g(x) = x - f(x)/f'(x)$ as in Newton's method and show that

$$\lim_{x\to\xi} g'(x) = \frac{m-1}{m} \qquad \text{and} \qquad \lim_{x\to\xi} h'(x) = 0.$$

Conclude that even when $f'(x) = 0$ this method, unlike Newton's method, has an accelerating rate of convergence as x_n approaches the solution $x = \xi$ to $f(x) = 0$.

f. Define

$$x_{n+1} = \frac{x_n + 2x_{n-1} + x_{n-2}}{4} - \frac{F_n F_n'}{[F_n']^2 - F_n F_n''}$$

where

$$\begin{aligned} F_n &= f\Big(\frac{x_n + 2x_{n-1} + x_{n-2}}{4}\Big), \\ F_n' &= \frac{2}{x_n - x_{n-2}}\Big\{f\Big(\frac{x_n + x_{n-1}}{2}\Big) - f\Big(\frac{x_{n-1} + x_{n-2}}{2}\Big)\Big\}, \\ F_n'' &= \frac{2}{x_n - x_{n-2}}\Big(\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} - \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}\Big) \end{aligned}$$

to create a secant-method-like approximation of the method given by h that doesn't involve f' and f''. Study this method both numerically and analytically. Test this method for the functions $f(x) = x^2 - 2$ and

$$f(x) = 2\cos(5x) + 2\cos(4x) + 6\cos(3x) + 4\cos(2x) + 10\cos(x) + 3.$$

How does this method compare to the secant method?