Hermite Interpolation Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Let x_1 and x_2 be distinct. The corresponding Lagrange basis functions are given by

$$\ell_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad \text{and} \qquad \ell_2(x) = \frac{x - x_1}{x_2 - x_1}.$$

Consider the following polynomials $\eta_1(x) = \ell_1(x)^2\ell_2(x)$ and $\eta_2(x) = \ell_1(x)\ell_2(x)^2$. Use calculus to find $\eta_i(x_j)$ and $\eta_i'(x_j)$ for $i = 1, 2$ and $j = 1, 2$.

b. Let A and B be constants and define $g_1(x) = A\eta_1(x)$ and $g_2(x) = B\eta_2(x)$. Solve for A and B such that

$$g_1(x_1) = 0, \quad g_1(x_2) = 0, \quad g_1'(x_1) = 1 \quad \text{and} \quad g_1'(x_2) = 0$$

and

$$g_2(x_1) = 0, \quad g_2(x_2) = 0, \quad g_2'(x_1) = 0 \quad \text{and} \quad g_2'(x_2) = 1.$$

c. Let a and b be constants and consider the polynomials

$$h_1(x) = \ell_1(x) + a\big(g_1(x) + g_2(x)\big) \quad \text{and} \quad h_2(x) = \ell_2(x) + b\big(g_1(x) + g_2(x)\big).$$

Solve for a and b such that

$$h_1(x_1) = 1, \quad h_1(x_2) = 0, \quad h_1'(x_1) = 0 \quad \text{and} \quad h_1'(x_2) = 0$$

and

$$h_2(x_1) = 0, \quad h_2(x_2) = 1, \quad h_2'(x_1) = 0 \quad \text{and} \quad h_2'(x_2) = 0.$$

d. Let y_i and z_i for $i = 1, 2$ be constants. Show that the polynomial given by

$$p(x) = \sum_{i=1}^{2} \Big\{ y_i h_i(x) + z_i g_i(x) \Big\}$$

satisfies the conditions $p(x_i) = y_i$ and $p'(x_i) = z_i$ for $i = 1, 2$ and explain why $p(x)$ is the unique polynomial of minimal degree that satisfies those conditions.

e. Let $f(x) = e^x - 13/x$ and define $y_i = f(x_i)$ and $z_i = f'(x_i)$ for $i = 1, 2$ where $x_1 = 1.3$ and $x_2 = 2.7$. Write a computer program that computes $p(x)$ for any value of x, verify $p(2) \approx 0.9393631693832503$ and then compute the size of the error $e = |f(2) - p(2)|$.

f. Plot $f(x)$ and $p(x)$ on the same graph over the interval $[0.5, 4]$. You might find the plotting commands

```
xs=[0.5:0.1:4;];
yf=f.(xs);
yp=p.(xs);
plot(xs,[yf yp],size=[400,300],
    legend=:bottomright,label=["f" "p"])
```

to be useful when making the graph in Julia.