Poisson Programming Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Write a program to find the numerical solution to the Poisson equation with Dirichlet boundary conditions

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

using the five-point finite difference formula

$$u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = h^2 f_{i,j}$$

and Gauss-Seidel iteration.

b. Note that if $\Delta\phi = f$ then $u = \phi$ is the exact solution on all of Ω. Using this observation, test your program for $f = 0$ and $\phi(x, y) = x - 2y$ on the domain Ω whose boundary $\partial\Omega$ consists of the triangle whose vertices are $(0, 0)$, $(7, 0)$ and $(0, 7)$. Try values of h equal to 2, 1, 1/2, 1/4 and 1/8.

c. How do the approximations of u found in the previous step depend on grid spacing? Explain your results in terms of the order of accuracy in the five-point formula and the specific form of ϕ used.

d. Now consider $f(x, y) = \sin\big(xy(x + y - 7)\big)$ and $\phi = 0$. Let U_h be the approximation of $u(2, 2)$ obtained using a grid spacing of size h. Plot $\log|U_h - U_{h/2}|$ versus h for values of h between 2 and 1/32. Verify the order of the numerical method.

e. Estimate the value of $u(2, 2)$ accurately to 3 decimal places.

f. Modify your program to use successive over-relaxation. Determine the optimum value of the relaxation parameter ω for a grid spacing of size $h = 1/32$. Does the optimal value of ω depend on f or ψ? How does the optimal ω depend on h?