The Polar Decomposition Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Let $A \in \mathbf{R}^{n\times n}$ and consider the iteration

$$X_{k+1} = \frac{1}{2}\Big(X_k + \big(X_k^{-1}\big)^T\Big) \qquad \text{where} \qquad X_0 = A.$$

Write a program to perform this iteration and test your program with the input

$$A = \begin{bmatrix} -0.49 & -0.21 & -0.40 & -0.21 \\ 0.36 & 0.10 & 0.29 & -0.04 \\ 0.12 & -0.01 & 0.48 & -0.47 \\ 0.09 & 0.09 & -0.41 & 0.22 \end{bmatrix}.$$

Verify the Frobenius matrix norm $\|X_0 - X_1\|_F \approx 16.37054203598731$.

b. Define $\Delta_k = X_{k+1} - X_k$. If X_k converges then it follows that $\Delta_k \to 0$ as $k \to \infty$. Compute $\|\Delta_k\|_F$ for $k = 0, \ldots, 9$.

c. Suppose X_k is α-order convergent such that $\|\Delta_{k+1}\|_F \approx M\|\Delta_k\|_F^{\alpha}$. Then

$$\log \|\Delta_{k+1}\|_F \approx \log M + \alpha \log \|\Delta_k\|_F$$

would show $\log \|\Delta_{k+1}\|_F$ is a linear function of $\log \|\Delta_k\|_F$. Plot the points

$$(\log \|\Delta_k\|_F, \log \|\Delta_{k+1}\|_F) \qquad \text{for} \qquad k = 1, \ldots, 8.$$

Do the points fall on a line? Find the slope between the last two points by computing

$$\alpha \approx \frac{\log \|\Delta_9\|_F - \log \|\Delta_8\|_F}{\log \|\Delta_8\|_F - \log \|\Delta_7\|_F}.$$

d. Let W be the limit of X_k as $k \to \infty$. Numerically check whether W is an orthogonal matrix by computing $X_k^T X_k$ for $k = 8, 9, 10$. What are your conclusions?

e. Define $P = W^{-1}A$ so that $A = WP$. This is called the polar decomposition of A. Use the built-in Julia function `eigvals` to find the eigenvalues of P and A. Are the eigenvalues of A positive? What about the eigenvalues of P?

f. Research the polar decomposition online or in a book on linear algebra. Consider the singular value decomposition $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal. Is it true or false that $W = UV^T$.