Viscous Burgers' Equation Project

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output.

1a. Consider the viscous Burgers equation

$$u_t - u_{xx} + 2uu_x = 0 \quad \text{where } x \in [0, 2\pi] \text{ and } t \geq 0$$

with initial condition $u(x, 0) = u_0(x)$ and periodic boundary conditions $u(0, t) = u(2\pi, t)$. Derive the semi-discrete spectral method

$$\frac{d\hat{u}_k}{dt} + k^2\hat{u}_k + \sum_{m=-\min(n,n-k)}^{\min(n,n+k)} ik\hat{u}_{k-m}\hat{u}_m = 0$$

by making the the approximation

$$u(x, t) \approx \sum_{k=-n}^{n} \hat{u}_k(t)e^{ikx}.$$

b. Find a subroutine for performing a fast Fourier transform. Connect to the National Institute of Standards and Technology web page at

`http://math.nist.gov/`

From this page select the following links in order:

1. Guide to Available Mathematical Software
2. What Problem it Solves
3. Taxonomy as a Decision Tree
4. J Integral Transforms
5. J1 Trigonometric Transforms
6. J1a One-dimensional
7. J1a2 Complex

How many routines are there to perform a fast Fourier transform? Download one of them and print it out.

c. Let $\mathcal{F}$ denote the Fourier transform on the interval $[0, 2\pi]$. The convolution theorem may be used to obtain that

$$\sum_{m=-\min(n,n-k)}^{\min(n,n+k)} ik\hat{u}_{k-m}\hat{u}_m = ik\mathcal{F}\{(\mathcal{F}^{-1}\hat{u})^2\}_k.$$

Compare the number operations needed to evaluate the sum directly with the number of operations needed to evaluate it by means of a fast Fourier transform. Note that the Fourier transform method obtains the sum for all values of k simultaneously. How does the total computational cost of each method depend on n?

d. Write a program to solve

$$\frac{d\hat{u}_k}{dt} + k^2\hat{u}_k + ik\mathcal{F}\{(\mathcal{F}^{-1}\hat{u})^2\}_k = 0$$

using Euler's explicit method. Let $\hat{u}_k(0) = \mathcal{F}\{8e^{(\pi-x)^2}\}_k$ and draw accurate graphs of $u(x, 1/8)$, $u(x, 1/4)$, $u(x, 1/2)$ and $u(x, 1)$.

e. Consider the semi-discrete central difference method

$$\frac{du_j}{dt} - \frac{1}{(\Delta x)^2}(u_{j-1} - 2u_j + u_{j+1}) + \frac{1}{\Delta x}u_j(u_{j+1} - u_{j-1}) = 0.$$

By taking $u_n = u_0$ and $u_{-1} = u_{n-1}$ this method may also be used to compute periodic solutions. Write a program using Euler's explicit method to solve these equations. Let $u(x, 0) = 8e^{(\pi-x)^2}$ on $[0, 2\pi]$ and compare the graphs of $u(x, 1/8)$, $u(x, 1/4)$, $u(x, 1/2)$ and $u(x, 1)$ with those found in the previous step.

f. Replace Euler's explicit method for the time integration by the explicit Runga-Kutta method of order four given by

$$\begin{aligned} K_0 &= \Delta t\, F(u^m) \\ K_1 &= \Delta t\, F(u^m + K_0/2) \\ K_2 &= \Delta t\, F(u^m + K_1/2) \\ K_3 &= \Delta t\, F(u^m + K_2) \\ u^{m+1} &= u^m + \frac{1}{6}(K_0 + 2K_1 + 2K_2 + K_3) \end{aligned}$$

and find $u(3\pi/2, 1/8)$ accurately to 3 decimal places.