Three Stage Runge–Kutta Methods

1a. The Shu–Osher TVD Runge–Kutta scheme given by the tableau

\[
\begin{array}{c|ccc}
0 & 1 & 1 & 2 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\
\hline
\frac{1}{6} & \frac{1}{6} & \frac{2}{3} & \\
\end{array}
\]

Find the truncation error and order for this method.

1b. Use the Shu–Osher TVD Runge–Kutta scheme to approximate the solution to

\[y' = y^2 \cos(t), \quad y(0) = 0.8\]

on the interval \([0, 8]\). Graph your solution.

1c. The exact solution to this equation is

\[y(t) = \frac{y_0}{1 - y_0 \sin(t)}.\]

Let \(y_n\) be the approximation of \(y(8)\) obtained by the Shu–Osher TVD Runge–Kutta scheme using \(n\) equal steps of size \(h = 8/n\). Graph \(\log |y_n - y(8)|\) versus \(\log h\) to verify the order of convergence found in part 1a numerically.

1d. [Extra Credit and for Math/CS 667] The classical Runge–Kutta scheme and the Nystrom Runge–Kutta schemes are given by

\[
\begin{array}{c|ccc}
0 & 1 & 1 & 2 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\
\hline
\frac{1}{6} & \frac{1}{6} & \frac{2}{3} & \\
\end{array}
\]

\[
\begin{array}{c|ccc}
0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{3} \\
\hline
\frac{1}{4} & \frac{3}{8} & \frac{3}{8} & \\
\end{array}
\]

respectively. Let \(z_n\) be the approximation of \(y(8)\) obtained from the classical RK scheme and \(w_n\) be obtained from the Nystrom RK scheme using \(n\) equal steps of size \(h = 8/n\). Compare \(\log |z_n - y(8)|\) and \(\log |w_n - y(8)|\) to the values of \(\log |y_n - y(8)|\) for \(n = 50\) and \(n = 100\). Which scheme is preferrable when solving the equation in part 1b?