Efficiency of ERK Methods versus AB Methods

2a. Let $W_{\mathrm{rk}}(n)$ be the number of evaluations of $f(t,y)$ necessary for n steps of the classical Runge-Kutta scheme. Find an explicit formula for $W_{\mathrm{rk}}(n)$ and write an efficient code implementing this scheme.

2b. In a practical implementation of the 3-step Adams–Bashforth method, one could first approximate y_1 and y_2 using the classical Runge–Kutta scheme and then compute y_{n+1} for $n \geq 2$ using

$$y_{n+1} = y_n + \frac{h}{12}\big(23f(t_n,y_n) - 16f(t_{n-1},y_{n-1}) + 5f(t_{n-2},y_{n-2})\big).$$

Use the techniques of the previous programming assignment to verify numerically that this method is third order.

2c. Let $W_{\mathrm{ab}}(n)$ be the number of evaluations of $f(t,y)$ necessary for n steps of the 3-step Adams–Bashforth scheme described above. Find an explicit formula for $W_{\mathrm{ab}}(n)$.

2d. Let $\mathrm{CPU}_{\mathrm{rk}}(n)$ be the actual CPU time needed to perform n steps of the classical Runge–Kutta method and let $\mathrm{CPU}_{\mathrm{ab}}(n)$ be the time needed for the 3-step Adams–Bashforth method. Check whether the actual time needed to approximate the initial value problem

$$y' = y^2\cos(t), \qquad y(0) = 0.8$$

on the interval $[0,8]$ is proportional to the number of function evaluations of $f(t,y)$. In particular, to what extent are the ratios

$$\frac{\mathrm{CPU}_{\mathrm{rk}}(n)}{W_{\mathrm{rk}}(n)} \quad \text{and} \quad \frac{\mathrm{CPU}_{\mathrm{ab}}(n)}{W_{\mathrm{ab}}(n)}$$

independent of n and equal?

2e. Let x_n be the approximation to $y(8)$ obtained by the classical Runge–Kutta scheme using n equal steps of size $h = 8/n$ and let y_n be the approximation obtained by using the 3-step Adams–Bashforth method. Plot $\log(|x_n - y(8)|)$ versus $\log(W_{\mathrm{rk}}(n))$ and then and $\log(|y_n - y(8)|)$ versus $\log(W_{\mathrm{ab}}(n))$ on the same graph. Which scheme is more efficient?

2f. [Extra Credit and for Math/CS 667] Repeast 2d using the same computer with an optimizing compiler for a traditional programming language. Did the constant of proportionality change?