INSTRUCTIONS: This is a take-home exam. Do not work in groups. You may contact or ask your instructor for help and clarification. Do not ask other persons for help. Part (iii) of each question is extra credit for undergraduates; graduate students must answer part (iii) from 2 questions for full credit.

1. Consider the initial value problem

$$\begin{cases} y' = \dfrac{2xy}{x^2+y^2} \\ y(0) = 2. \end{cases}$$

(i) Write a program that uses the RK4 method to solve this problem and graph the solution $y(t)$ on the interval $[0, 2]$.

(ii) Find $y(2)$ to 5 significant digits.

(iii) [$*$] Find an exact analytic solution to this equation by making the substitution $v = y/x$ and solving the resulting equation using separation of variables.

2. Consider the heat equation on the domain $[0, 2]$ given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{3}\dfrac{\partial^2 u}{\partial x^2} \quad \text{for} \quad x \in (0, 2) \\ u(x, 0) = 0 \\ u(0, t) = 0 \\ u(2, t) = \sin(t^2). \end{cases}$$

(i) Use the backwards Euler method to approximate the solution $u(x, 3)$ to this equation and plot your approximation $u(x, 3)$ for $x \in [0, 2]$.

(ii) Compute $u(1, 3)$ to 2 significant digits.

(iii) [$*$] Starting with the semidiscrete equation

$$\frac{du}{dt} = Au + F$$

where

$$A = \frac{1/3}{h_x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \quad \text{and} \quad F = \frac{1/3}{h_x^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sin(t^2) \end{bmatrix}$$

Use RK2 to discretize in time and compute $u(1, 3)$ to 3 significant digits. Make sure to demonstrate your solution is converging.

3. Consider the Poisson equation on the domain Ω whose boundary $\partial\Omega$ consists of the triangle whose vertices are $(-5,-5)$, $(0,5)$ and $(5,0)$ given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{on } \Omega \\ u = \cos x - 2\sin y & \text{on } \partial\Omega. \end{cases}$$

(i) Use the five-point finite difference formula

$$u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1} = 0$$

and Gauss-Seidel iteration to approximate a solution to this equation using grid sizes of h equal to 2, 1, 1/2, 1/4 and 1/8.

(ii) Find the value of $u(0,0)$ when $h = 1/8$.

(iii) [*] Modify your program to use successive over-relaxation (SOR) instead of Gauss-Seidel. Determine the optimum value of the relaxation parameter ω for a grid spacing of size $h = 1/8$. See

`http://en.wikipedia.org/wiki/Successive_over-relaxation`

for more information.

4. Consider the linear model

$$F(x) = \sum_{i=0}^{3} c_i \cos(x\sqrt{i})$$

and the data points (x, y) given by

X	Y	X	Y	X	Y
1.3393	3.2455	-1.2786	22.4212	-2.5529	9.0218
5.0122	18.8167	3.4174	10.8772	1.4918	17.7334
0.6244	-2.3164	2.4482	18.5774	-0.8224	4.0931
5.1112	22.7087	-2.3399	15.1236	3.2268	11.5728
6.1239	-9.1789	-3.3490	-5.5905	0.6507	-2.6660

(i) Find the parameters c_i that give the best fit in the least-squares sense of the model to the data.

(ii) Plot the data and the model on the same graph.

(iii) [*] Comment on the goodness of fit.