§8.2#1b

Solve the problem

$$\begin{cases} y' = \dfrac{1}{1+x^2} - 2y^2, & 0 \le x \le 10 \\ y(0) = 0 \end{cases}$$

using Euler's method with stepsizes of $h = 0.2$, 0.1 and 0.05. Compute the error and relative error using the true answer $y(x) = x/(1+x^2)$. For selected values of x observe the ratio by which the error decreases when h is halved.

Euler's method is given by the code

```
function [x,y]=euler(x0,y0,xn,f,N)
  h=(xn-x0)/N;
  xn=x0;
  yn=y0;
  for n=1:N
    yn=yn+h*f(xn,yn);
    xn=x0+h*n;
    x(:,n)=xn;
    y(:,n)=yn;
  end;
```

the function `testit` is given by

```
function testit(fexact,solver,x0,y0,xt,f,hn)
    xn=[1:xt];
    for i=1:length(hn)
        h=hn(i);
        n=floor((xt-x0)/h+0.1);
        [x,y]=solver(x0,y0,xt,f,n);
        yexact=fexact(x);
        for j=1:length(xn)
            k=floor((xn(j)-x0)/h+0.1);
            yh(i,j)=y(k);
            eabs(i,j)=yexact(k)-y(k);
            erel(i,j)=eabs(i,j)/yexact(k);
        end
    end

    disp('Relative and Absolute Errors');
    disp(sprintf('%5s %5s %12s %12s %12s %12s',...
        'h','x','yh','error','rel-error'));
    for i=1:length(hn)
        disp('-----------------------------------------------------');
        for j=1:length(xn)
            disp(sprintf('%5g %5g %12g %12g %12g',...
                hn(i),xn(j),yh(i,j),eabs(i,j),erel(i,j)));
        end
```

§8.2#1b

```
    end

    disp('');
    disp('Ratios of Relative and Absolute Errors');
    disp(sprintf('%5s %5s %4s %16s %16s',...
        'h1','h2','x','abs-err-ratio','rel-err-ratio'));
    for i=2:length(hn)
        disp('------------------------------------------------');
        for j=1:length(xn)
            disp(sprintf('%5g %5g %4g %16g %16g',...
                hn(i-1),hn(i),xn(j),...
                eabs(i,j)/eabs(i-1,j),erel(i,j)/erel(i-1,j)));
        end
    end
end
```

which is called by the script

```
clear all
f=@(x,y) 1/(1+x^2)-2*y^2;
fexact=@(x) x./(1+x.^2);
x0=0; y0=0; xt=10;
hn=[0.2,0.1,0.05];
testit(fexact,@euler,x0,y0,xt,f,hn);
```

and gives the output

```
Relative and Absolute Errors
    h     x           yh         error     rel-error
----------------------------------------------------
  0.2     1     0.546605    -0.0466046    -0.0932093
  0.2     2     0.406819   -0.00681903    -0.0170476
  0.2     3     0.298894    0.00110632    0.00368773
  0.2     4     0.233156    0.00213804    0.00908668
  0.2     5     0.190322    0.00198534     0.0103238
  0.2     6     0.160498    0.00166446     0.0102642
  0.2     7      0.13863    0.00136988    0.00978483
  0.2     8     0.121946    0.00113095    0.00918894
  0.2     9     0.108814    0.00094258    0.00858795
  0.2    10    0.0982156   0.000794297     0.0080224
----------------------------------------------------
  0.1     1     0.522675    -0.0226754    -0.0453508
  0.1     2     0.404189   -0.00418867    -0.0104717
  0.1     3      0.29977    0.00023007     0.0007669
  0.1     4     0.234347   0.000947176     0.0040255
  0.1     5     0.191363   0.000944983    0.00491391
  0.1     6     0.161349   0.000813548    0.00501688
  0.1     7     0.139322   0.000678361    0.00484544
  0.1     8     0.122513    0.00056416     0.0045838
  0.1     9     0.109284    0.00047228    0.00430299
  0.1    10    0.0986108   0.000399096    0.00403087
```

§**8.2#1b**

0.05	1	0.511202	-0.0112025	-0.0224049
0.05	2	0.402271	-0.00227141	-0.00567852
0.05	3	0.299968	3.16855e-05	0.000105618
0.05	4	0.234854	0.000439654	0.00186853
0.05	5	0.191849	0.00045826	0.00238295
0.05	6	0.161761	0.000400678	0.00247085
0.05	7	0.139663	0.000336648	0.00240463
0.05	8	0.122796	0.000281183	0.00228462
0.05	9	0.10952	0.000236012	0.00215034
0.05	10	0.0988101	0.00019978	0.00201778

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.2	0.1	1	0.486548	0.486548
0.2	0.1	2	0.614261	0.614261
0.2	0.1	3	0.20796	0.20796
0.2	0.1	4	0.443011	0.443011
0.2	0.1	5	0.47598	0.47598
0.2	0.1	6	0.488775	0.488775
0.2	0.1	7	0.495199	0.495199
0.2	0.1	8	0.498839	0.498839
0.2	0.1	9	0.50105	0.50105
0.2	0.1	10	0.502452	0.502452
0.1	0.05	1	0.494036	0.494036
0.1	0.05	2	0.542275	0.542275
0.1	0.05	3	0.137721	0.137721
0.1	0.05	4	0.464173	0.464173
0.1	0.05	5	0.48494	0.48494
0.1	0.05	6	0.492506	0.492506
0.1	0.05	7	0.496267	0.496267
0.1	0.05	8	0.498411	0.498411
0.1	0.05	9	0.49973	0.49973
0.1	0.05	10	0.500582	0.500582

Observe that the errors decrease by half when h decreases by half.

§8.4#7

Solve the equation

$$\begin{cases} y' = \lambda y + \dfrac{1}{1+x^2} - \lambda \arctan x \\ y(0) = 0 \end{cases}$$

using Euler's method, the backward Euler method and the trapezoid method. The true solution is $y(x) = \arctan x$. Let $\lambda = -1, -10$ and -50; let $h = 0.5, 0.1$ and 0.001.

Since the forcing is linear in y we may solve the implicit system explicitly for the backwards Euler and trapezoidal methods. This needs to be done since the predictor-corrector iterations given in the text on page 399 are not absolutely stable. We create the backward Euler and trapezoidal integrators using Maple and include the machine optimized calculations directly into the routime. Thus, the argument `f` in these routines is ignored but included so the calling syntax is still compatible with the `testit` function given in the first question. The Maple script to generate the backward Euler's method is

```
restart;
with(codegen):
f:=(x,y)->lambda*y+1/(1+x^2)-lambda*arctan(x);
eq:=ynp1=yn+h*f(xn,ynp1);
beul:=solve(eq,ynp1);
beulo:=optimize(['yn'=beul]);
C([beulo]);
```

and backward Euler's method is given by

```
function [x,y]=beul84p7(x0,y0,xn,f,N)
  global lambda
  h=(xn-x0)/N;
  xn=x0;
  yn=y0;
  for n=1:N
    xn=x0+h*n;

% WARNING: The argumant f is not used; instead we directly
% insert the following block generated by Maple
      t1 = xn*xn;
      t3 = h*lambda;
      t4 = atan(xn);
      yn = -(yn+yn*t1+h-t3*t4-t3*t4*t1)/(-1.0-t1+t3+t3*t1);

    x(:,n)=xn;
    y(:,n)=yn;
  end;
```

The Maple script to generate the trapezoidal method is

```
restart;
with(codegen):
```

§**8.4#7**

```
f:=(x,y)->lambda*y+1/(1+x^2)-lambda*arctan(x);
eq:=ynp1=yn+h*(fxnyn+f(xn,ynp1))/2;
trap:=solve(eq,ynp1);
trapo:=optimize(['yn'=trap]);
C([trapo]);
```

and the trapezoidal method is given by

```
function [x,y]=trap84p7(x0,y0,xn,f,N)
  global lambda
  h=(xn-x0)/N;
  xn=x0;
  yn=y0;
  for n=1:N
    fxnyn=f(xn,yn);
    xn=x0+h*n;

% WARNING: The argument f is not used; instead we directly
% insert the following block generated by Maple
      t2 = xn*xn;
      t5 = h*fxnyn;
      t7 = h*lambda;
      t8 = atan(xn);
      yn = (-2.0*yn-2.0*yn*t2-t5-t5*t2-h+t7*t8+t7*t8*t2)/...
           (-2.0-2.0*t2+t7+t7*t2);

    x(:,n)=xn;
    y(:,n)=yn;
  end;
```

These routeines are called by the script

```
clear all
more off
global lambda
x0=0; y0=0; xt=10;
hn=[0.5,0.1,0.001];
lambdan=[-1,-10,-50];
fexact=@(x) atan(x);

function r=f(x,y)
    global lambda
    r=lambda*y+1/(1+x^2)-lambda*atan(x);
end

for i=1:length(lambdan)
    lambda=lambdan(i);
```

§8.4#7

```

    disp('');
    disp(sprintf('Eulers Explicit Method (lambda=%g)',lambda));
    testit(fexact,@euler,x0,y0,xt,@f,hn);

    disp('');
    disp(sprintf('Backwards Euler Method (lambda=%g)',lambda));
    testit(fexact,@beul84p7,x0,y0,xt,@f,hn);

    disp('');
    disp(sprintf('Trapezoidal Method (lambda=%g)',lambda));
    testit(fexact,@trap84p7,x0,y0,xt,@f,hn);
end
```

which gives the output

```
Eulers Explicit Method (lambda=-1)
Relative and Absolute Errors
    h     x           yh        error    rel-error
--------------------------------------------------
  0.5     1     0.881824   -0.0964256    -0.122773
  0.5     2      1.18705   -0.0798998   -0.0721672
  0.5     3      1.28766    -0.038614   -0.0309148
  0.5     4      1.34316   -0.0173429   -0.0130809
  0.5     5      1.38154  -0.00814278  -0.00592892
  0.5     6      1.40982  -0.00417523  -0.00297032
  0.5     7      1.43126  -0.00235931  -0.00165114
  0.5     8       1.4479  -0.00145415  -0.00100533
  0.5     9       1.4611 -0.000960982 -0.000658144
  0.5    10       1.4718 -0.000670051 -0.000455467
--------------------------------------------------
  0.1     1     0.802576   -0.0171779   -0.0218716
  0.1     2      1.12215   -0.0150026   -0.0135506
  0.1     3      1.25732   -0.0082718  -0.00662249
  0.1     4      1.32997  -0.00415467  -0.00313367
  0.1     5      1.37548  -0.00208342  -0.00151698
  0.1     6      1.40673  -0.00108551 -0.000772248
  0.1     7       1.4295 -0.000600321 -0.000420128
  0.1     8       1.4468 -0.000355598 -0.000245843
  0.1     9      1.46036  -0.00022539 -0.000154362
  0.1    10      1.47128 -0.000151705 -0.000103121
--------------------------------------------------
0.001     1     0.785566 -0.000167715 -0.000213542
0.001     2       1.1073 -0.000147901 -0.000133587
0.001     3      1.24913 -8.35313e-05 -6.68761e-05
0.001     4      1.32586 -4.29257e-05 -3.23768e-05
0.001     5      1.37342 -2.18965e-05 -1.59433e-05
0.001     6      1.40566 -1.15136e-05 -8.19094e-06
0.001     7      1.42891 -6.37457e-06 -4.46117e-06
0.001     8      1.44645  -3.7568e-06 -2.59727e-06
0.001     9      1.46014 -2.36134e-06  -1.6172e-06
0.001    10      1.47113  -1.5752e-06 -1.07075e-06
```

§8.4#7

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	0.178146	0.178146
0.5	0.1	2	0.187767	0.187767
0.5	0.1	3	0.214217	0.214217
0.5	0.1	4	0.23956	0.23956
0.5	0.1	5	0.255862	0.255862
0.5	0.1	6	0.259988	0.259988
0.5	0.1	7	0.254448	0.254448
0.5	0.1	8	0.244539	0.244539
0.5	0.1	9	0.234541	0.234541
0.5	0.1	10	0.226408	0.226408
0.1	0.001	1	0.00976345	0.00976345
0.1	0.001	2	0.00985834	0.00985834
0.1	0.001	3	0.0100983	0.0100983
0.1	0.001	4	0.0103319	0.0103319
0.1	0.001	5	0.0105099	0.0105099
0.1	0.001	6	0.0106066	0.0106066
0.1	0.001	7	0.0106186	0.0106186
0.1	0.001	8	0.0105648	0.0105648
0.1	0.001	9	0.0104767	0.0104767
0.1	0.001	10	0.0103834	0.0103834

Backwards Euler Method (lambda=-1)
Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	0.709277	0.0761215	0.0969209
0.5	2	1.03772	0.0694237	0.062705
0.5	3	1.20605	0.0429925	0.0344203
0.5	4	1.30156	0.024254	0.0182936
0.5	5	1.36002	0.013377	0.00974004
0.5	6	1.39822	0.00742489	0.00528218
0.5	7	1.42468	0.00421865	0.00295238
0.5	8	1.44396	0.00248306	0.00171667
0.5	9	1.45861	0.00152636	0.00104535
0.5	10	1.47014	0.000983947	0.000668838
0.1	1	0.769003	0.0163951	0.0208749
0.1	2	1.09256	0.0145849	0.0131733
0.1	3	1.24062	0.00842175	0.00674255
0.1	4	1.3214	0.00442247	0.00333565
0.1	5	1.37111	0.00229457	0.00167072
0.1	6	1.40443	0.00121898	0.0008672
0.1	7	1.42822	0.0006769	0.000473721
0.1	8	1.44604	0.000397624	0.000274898
0.1	9	1.45989	0.000248132	0.000169937
0.1	10	1.47096	0.000164105	0.00011155
0.001	1	0.785231	0.000167637	0.000213442
0.001	2	1.107	0.000147859	0.000133549
0.001	3	1.24896	8.35462e-05	6.68881e-05

§8.4#7

0.001	4	1.32577	4.29525e-05	3.2397e-05
0.001	5	1.37338	2.19176e-05	1.59587e-05
0.001	6	1.40564	1.15269e-05	8.20045e-06
0.001	7	1.42889	6.38223e-06	4.46654e-06
0.001	8	1.44644	3.76101e-06	2.60018e-06
0.001	9	1.46014	2.36362e-06	1.61876e-06
0.001	10	1.47113	1.57644e-06	1.07159e-06

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	0.21538	0.21538
0.5	0.1	2	0.210084	0.210084
0.5	0.1	3	0.195889	0.195889
0.5	0.1	4	0.18234	0.18234
0.5	0.1	5	0.171531	0.171531
0.5	0.1	6	0.164174	0.164174
0.5	0.1	7	0.160454	0.160454
0.5	0.1	8	0.160135	0.160135
0.5	0.1	9	0.162564	0.162564
0.5	0.1	10	0.166782	0.166782
0.1	0.001	1	0.0102248	0.0102248
0.1	0.001	2	0.0101378	0.0101378
0.1	0.001	3	0.00992029	0.00992029
0.1	0.001	4	0.00971233	0.00971233
0.1	0.001	5	0.00955196	0.00955196
0.1	0.001	6	0.00945624	0.00945624
0.1	0.001	7	0.00942862	0.00942862
0.1	0.001	8	0.00945871	0.00945871
0.1	0.001	9	0.00952565	0.00952565
0.1	0.001	10	0.00960633	0.00960633

Trapezoidal Method (lambda=-1)
Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	0.781447	0.0039513	0.00503095
0.5	2	1.10995	-0.00280521	-0.00253372
0.5	3	1.25128	-0.00223536	-0.00178965
0.5	4	1.32702	-0.00120715	-0.000910496
0.5	5	1.374	-0.000597156	-0.000434801
0.5	6	1.40594	-0.000291719	-0.000207534
0.5	7	1.42904	-0.0001456	-0.000101896
0.5	8	1.44652	-7.57777e-05	-5.2389e-05
0.5	9	1.46018	-4.16402e-05	-2.8518e-05
0.5	10	1.47115	-2.42879e-05	-1.65097e-05
0.1	1	0.78526	0.000137955	0.000175649
0.1	2	1.10726	-0.000113112	-0.000102165
0.1	3	1.24914	-8.92434e-05	-7.14492e-05
0.1	4	1.32587	-4.84906e-05	-3.65741e-05
0.1	5	1.37342	-2.41733e-05	-1.7601e-05
0.1	6	1.40566	-1.18871e-05	-8.45665e-06
0.1	7	1.42891	-5.9585e-06	-4.16999e-06

§8.4#7

```
  0.1     8      1.44644 -3.10593e-06 -2.14729e-06
  0.1     9      1.46014 -1.70501e-06  -1.1677e-06
  0.1    10      1.47113 -9.91695e-07 -6.74105e-07
-------------------------------------------------
0.001     1     0.785398  1.37206e-08  1.74696e-08
0.001     2      1.10715 -1.13133e-08 -1.02184e-08
0.001     3      1.24905 -8.92315e-09 -7.14397e-09
0.001     4      1.32582 -4.84967e-09 -3.65787e-09
0.001     5       1.3734 -2.41847e-09 -1.76094e-09
0.001     6      1.40565 -1.18962e-09 -8.46317e-10
0.001     7       1.4289 -5.96393e-10 -4.17379e-10
0.001     8      1.44644  -3.1088e-10 -2.14928e-10
0.001     9      1.46014 -1.70649e-10 -1.16872e-10
0.001    10      1.47113 -9.91687e-11   -6.741e-11

Ratios of Relative and Absolute Errors
   h1    h2     x    abs-err-ratio    rel-err-ratio
-------------------------------------------------
  0.5   0.1     1        0.0349138        0.0349138
  0.5   0.1     2        0.0403223        0.0403223
  0.5   0.1     3        0.0399236        0.0399236
  0.5   0.1     4        0.0401695        0.0401695
  0.5   0.1     5        0.0404807        0.0404807
  0.5   0.1     6        0.0407483        0.0407483
  0.5   0.1     7        0.0409239        0.0409239
  0.5   0.1     8        0.0409874        0.0409874
  0.5   0.1     9        0.0409462        0.0409462
  0.5   0.1    10        0.0408309        0.0408309
-------------------------------------------------
  0.1 0.001     1      9.94574e-05      9.94574e-05
  0.1 0.001     2      0.000100018      0.000100018
  0.1 0.001     3      9.99867e-05      9.99867e-05
  0.1 0.001     4      0.000100013      0.000100013
  0.1 0.001     5      0.000100047      0.000100047
  0.1 0.001     6      0.000100077      0.000100077
  0.1 0.001     7      0.000100091      0.000100091
  0.1 0.001     8      0.000100092      0.000100092
  0.1 0.001     9      0.000100087      0.000100087
  0.1 0.001    10      9.99992e-05      9.99992e-05

Eulers Explicit Method (lambda=-10)
Relative and Absolute Errors
    h     x           yh        error    rel-error
-------------------------------------------------
  0.5     1     0.718238    0.0671601    0.0855109
  0.5     2     -0.14834      1.25549      1.13398
  0.5     3      -18.896       20.145      16.1283
  0.5     4     -321.017      322.342      243.127
  0.5     5     -5156.12      5157.49      3755.27
  0.5     6     -82518.4      82519.8      58705.9
  0.5     7 -1.32032e+06  1.32032e+06       924010
  0.5     8 -2.11251e+07  2.11251e+07  1.46049e+07
  0.5     9 -3.38001e+08  3.38001e+08  2.31486e+08
  0.5    10 -5.40802e+09  5.40802e+09  3.67611e+09
-------------------------------------------------
```

§**8.4#7**

```
  0.1     1     0.788064  -0.00266556  -0.00339389
  0.1     2      1.10801 -0.000861654 -0.000778264
  0.1     3      1.24936 -0.000318079 -0.000254658
  0.1     4      1.32596 -0.000145008 -0.000109372
  0.1     5      1.37348 -7.68521e-05 -5.59575e-05
  0.1     6      1.40569 -4.52702e-05  -3.2206e-05
  0.1     7      1.42893 -2.87951e-05 -2.01519e-05
  0.1     8      1.44646 -1.94072e-05 -1.34172e-05
  0.1     9      1.46015 -1.36823e-05 -9.37057e-06
  0.1    10      1.47114 -9.99932e-06 -6.79705e-06
--------------------------------------------------
0.001     1     0.785425 -2.72596e-05 -3.47081e-05
0.001     2      1.10716 -9.00796e-06 -8.13618e-06
0.001     3      1.24905 -3.29209e-06 -2.63568e-06
0.001     4      1.32582 -1.48905e-06 -1.12312e-06
0.001     5       1.3734 -7.85055e-07 -5.71614e-07
0.001     6      1.40565 -4.60779e-07 -3.27805e-07
0.001     7       1.4289 -2.92328e-07 -2.04583e-07
0.001     8      1.44644 -1.96639e-07 -1.35947e-07
0.001     9      1.46014 -1.38424e-07 -9.48018e-08
0.001    10      1.47113 -1.01041e-07 -6.86827e-08

Ratios of Relative and Absolute Errors
   h1    h2    x    abs-err-ratio    rel-err-ratio
--------------------------------------------------
  0.5   0.1    1       -0.0396896       -0.0396896
  0.5   0.1    2      -0.00068631      -0.00068631
  0.5   0.1    3     -1.57895e-05     -1.57895e-05
  0.5   0.1    4     -4.49857e-07     -4.49857e-07
  0.5   0.1    5     -1.49011e-08     -1.49011e-08
  0.5   0.1    6     -5.48598e-10     -5.48598e-10
  0.5   0.1    7     -2.18092e-11     -2.18092e-11
  0.5   0.1    8      -9.1868e-13      -9.1868e-13
  0.5   0.1    9     -4.04802e-14     -4.04802e-14
  0.5   0.1   10     -1.84898e-15     -1.84898e-15
--------------------------------------------------
  0.1 0.001    1        0.0102266        0.0102266
  0.1 0.001    2        0.0104543        0.0104543
  0.1 0.001    3        0.0103499        0.0103499
  0.1 0.001    4        0.0102688        0.0102688
  0.1 0.001    5        0.0102151        0.0102151
  0.1 0.001    6        0.0101784        0.0101784
  0.1 0.001    7         0.010152         0.010152
  0.1 0.001    8        0.0101323        0.0101323
  0.1 0.001    9         0.010117         0.010117
  0.1 0.001   10        0.0101048        0.0101048

Backwards Euler Method (lambda=-10)
Relative and Absolute Errors
    h     x           yh        error    rel-error
--------------------------------------------------
  0.5     1     0.771672    0.0137264     0.017477
  0.5     2       1.1015   0.00565016   0.00510334
  0.5     3      1.24704   0.00201002   0.00160925
  0.5     4      1.32495  0.000866124  0.000653275
```

§**8.4#7**

0.5	5	1.37296	0.000441726	0.000321629
0.5	6	1.40539	0.000253657	0.000180455
0.5	7	1.42874	0.000158499	0.000110924
0.5	8	1.44634	0.000105438	7.28949e-05
0.5	9	1.46007	7.35979e-05	5.04047e-05
0.5	10	1.47107	5.3366e-05	3.62756e-05
0.1	1	0.782635	0.00276275	0.00351764
0.1	2	1.1062	0.000943893	0.000852544
0.1	3	1.2487	0.000341522	0.000273426
0.1	4	1.32566	0.00015314	0.000115506
0.1	5	1.37332	8.0278e-05	5.8452e-05
0.1	6	1.4056	4.69359e-05	3.33909e-05
0.1	7	1.42887	2.96949e-05	2.07817e-05
0.1	8	1.44642	1.99335e-05	1.37811e-05
0.1	9	1.46013	1.40099e-05	9.5949e-06
0.1	10	1.47112	1.02135e-05	6.94262e-06
0.001	1	0.785371	2.72693e-05	3.47204e-05
0.001	2	1.10714	9.01619e-06	8.14361e-06
0.001	3	1.24904	3.29443e-06	2.63756e-06
0.001	4	1.32582	1.48986e-06	1.12373e-06
0.001	5	1.3734	7.85397e-07	5.71863e-07
0.001	6	1.40565	4.60945e-07	3.27924e-07
0.001	7	1.4289	2.92418e-07	2.04646e-07
0.001	8	1.44644	1.96691e-07	1.35983e-07
0.001	9	1.46014	1.38457e-07	9.48242e-08
0.001	10	1.47113	1.01062e-07	6.86973e-08

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	0.201272	0.201272
0.5	0.1	2	0.167056	0.167056
0.5	0.1	3	0.169909	0.169909
0.5	0.1	4	0.17681	0.17681
0.5	0.1	5	0.181737	0.181737
0.5	0.1	6	0.185037	0.185037
0.5	0.1	7	0.187351	0.187351
0.5	0.1	8	0.189054	0.189054
0.5	0.1	9	0.190357	0.190357
0.5	0.1	10	0.191386	0.191386
0.1	0.001	1	0.00987036	0.00987036
0.1	0.001	2	0.00955213	0.00955213
0.1	0.001	3	0.00964631	0.00964631
0.1	0.001	4	0.00972879	0.00972879
0.1	0.001	5	0.00978346	0.00978346
0.1	0.001	6	0.00982074	0.00982074
0.1	0.001	7	0.00984743	0.00984743
0.1	0.001	8	0.00986736	0.00986736
0.1	0.001	9	0.00988277	0.00988277
0.1	0.001	10	0.00989501	0.00989501

Trapezoidal Method (lambda=-10)

§8.4#7

Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	0.787998	-0.00259955	-0.00330985
0.5	2	1.10781	-0.000656816	-0.00059325
0.5	3	1.24921	-0.000164067	-0.000131354
0.5	4	1.32587	-5.13017e-05	-3.86944e-05
0.5	5	1.37342	-2.02113e-05	-1.47162e-05
0.5	6	1.40566	-9.57215e-06	-6.80978e-06
0.5	7	1.4289	-5.15601e-06	-3.60838e-06
0.5	8	1.44644	-3.0339e-06	-2.09749e-06
0.5	9	1.46014	-1.90277e-06	-1.30314e-06
0.5	10	1.47113	-1.25312e-06	-8.51809e-07
0.1	1	0.785436	-3.82376e-05	-4.86857e-05
0.1	2	1.10717	-1.68397e-05	-1.52099e-05
0.1	3	1.24905	-4.87912e-06	-3.90628e-06
0.1	4	1.32582	-1.75421e-06	-1.32311e-06
0.1	5	1.3734	-7.59077e-07	-5.52699e-07
0.1	6	1.40565	-3.76217e-07	-2.67647e-07
0.1	7	1.4289	-2.06136e-07	-1.44262e-07
0.1	8	1.44644	-1.21892e-07	-8.42704e-08
0.1	9	1.46014	-7.65015e-08	-5.23933e-08
0.1	10	1.47113	-5.03592e-08	-3.42317e-08
0.001	1	0.785398	-3.77416e-09	-4.80541e-09
0.001	2	1.10715	-1.68678e-09	-1.52354e-09
0.001	3	1.24905	-4.88759e-10	-3.91306e-10
0.001	4	1.32582	-1.75622e-10	-1.32463e-10
0.001	5	1.3734	-7.5981e-11	-5.53233e-11
0.001	6	1.40565	-3.76408e-11	-2.67782e-11
0.001	7	1.4289	-2.06235e-11	-1.44331e-11
0.001	8	1.44644	-1.2196e-11	-8.43174e-12
0.001	9	1.46014	-7.66343e-12	-5.24842e-12
0.001	10	1.47113	-5.03042e-12	-3.41943e-12

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	0.0147093	0.0147093
0.5	0.1	2	0.0256383	0.0256383
0.5	0.1	3	0.0297385	0.0297385
0.5	0.1	4	0.0341939	0.0341939
0.5	0.1	5	0.0375571	0.0375571
0.5	0.1	6	0.0393033	0.0393033
0.5	0.1	7	0.0399797	0.0399797
0.5	0.1	8	0.0401767	0.0401767
0.5	0.1	9	0.0402054	0.0402054
0.5	0.1	10	0.0401871	0.0401871
0.1	0.001	1	9.87028e-05	9.87028e-05
0.1	0.001	2	0.000100167	0.000100167
0.1	0.001	3	0.000100174	0.000100174
0.1	0.001	4	0.000100115	0.000100115
0.1	0.001	5	0.000100097	0.000100097

§8.4#7

0.1	0.001	6	0.000100051	0.000100051
0.1	0.001	7	0.000100048	0.000100048
0.1	0.001	8	0.000100056	0.000100056
0.1	0.001	9	0.000100174	0.000100174
0.1	0.001	10	9.98909e-05	9.98909e-05

Eulers Explicit Method (lambda=-50)
Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	-0.00880977	0.794208	1.01122
0.5	2	-457.59	458.697	414.305
0.5	3	-264208	264210	211529
0.5	4	-1.52185e+08	1.52185e+08	1.14786e+08
0.5	5	-8.76585e+10	8.76585e+10	6.38258e+10
0.5	6	-5.04913e+13	5.04913e+13	3.59203e+13
0.5	7	-2.9083e+16	2.9083e+16	2.03534e+16
0.5	8	-1.67518e+19	1.67518e+19	1.15814e+19
0.5	9	-9.64903e+21	9.64903e+21	6.6083e+21
0.5	10	-5.55784e+24	5.55784e+24	3.77795e+24
0.1	1	-27.7242	28.5096	36.2995
0.1	2	-2.9895e+07	2.9895e+07	2.70018e+07
0.1	3	-3.13472e+13	3.13472e+13	2.50969e+13
0.1	4	-3.28699e+19	3.28699e+19	2.47922e+19
0.1	5	-3.44666e+25	3.44666e+25	2.50958e+25
0.1	6	-3.61408e+31	3.61408e+31	2.57111e+31
0.1	7	-3.78964e+37	3.78964e+37	2.65214e+37
0.1	8	-3.97372e+43	3.97372e+43	2.74724e+43
0.1	9	-4.16675e+49	4.16675e+49	2.85367e+49
0.1	10	-4.36916e+55	4.36916e+55	2.96994e+55
0.001	1	0.785403	-5.09809e-06	-6.49108e-06
0.001	2	1.10715	-1.63552e-06	-1.47724e-06
0.001	3	1.24905	-6.10453e-07	-4.88736e-07
0.001	4	1.32582	-2.80647e-07	-2.11678e-07
0.001	5	1.3734	-1.4961e-07	-1.08934e-07
0.001	6	1.40565	-8.84966e-08	-6.29579e-08
0.001	7	1.4289	-5.64644e-08	-3.9516e-08
0.001	8	1.44644	-3.8146e-08	-2.63723e-08
0.001	9	1.46014	-2.69439e-08	-1.8453e-08
0.001	10	1.47113	-1.9721e-08	-1.34053e-08

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	35.8969	35.8969
0.5	0.1	2	65173.7	65173.7
0.5	0.1	3	1.18645e+08	1.18645e+08
0.5	0.1	4	2.15987e+11	2.15987e+11
0.5	0.1	5	3.93192e+14	3.93192e+14
0.5	0.1	6	7.15783e+17	7.15783e+17
0.5	0.1	7	1.30304e+21	1.30304e+21
0.5	0.1	8	2.37212e+24	2.37212e+24
0.5	0.1	9	4.31831e+27	4.31831e+27

§8.4#7

0.5	0.1	10	7.86125e+30	7.86125e+30
0.1	0.001	1	-1.7882e-07	-1.7882e-07
0.1	0.001	2	-5.47089e-14	-5.47089e-14
0.1	0.001	3	-1.9474e-20	-1.9474e-20
0.1	0.001	4	-8.53812e-27	-8.53812e-27
0.1	0.001	5	-4.34072e-33	-4.34072e-33
0.1	0.001	6	-2.44866e-39	-2.44866e-39
0.1	0.001	7	-1.48997e-45	-1.48997e-45
0.1	0.001	8	-9.59956e-52	-9.59956e-52
0.1	0.001	9	-6.4664e-58	-6.4664e-58
0.1	0.001	10	-4.51368e-64	-4.51368e-64

Backwards Euler Method (lambda=-50)
Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	0.782544	0.00285379	0.00363356
0.5	2	1.10614	0.00100537	0.000908075
0.5	3	1.24869	0.00035922	0.000287595
0.5	4	1.32566	0.000159341	0.000120183
0.5	5	1.37332	8.29108e-05	6.0369e-05
0.5	6	1.4056	4.82234e-05	3.43069e-05
0.5	7	1.42887	3.03935e-05	2.12706e-05
0.5	8	1.44642	2.03436e-05	1.40646e-05
0.5	9	1.46012	1.42658e-05	9.77016e-06
0.5	10	1.47112	1.03812e-05	7.0566e-06
0.1	1	0.784872	0.000526028	0.00066976
0.1	2	1.10698	0.000170108	0.000153645
0.1	3	1.24898	6.29537e-05	5.04015e-05
0.1	4	1.32579	2.87554e-05	2.16888e-05
0.1	5	1.37339	1.52613e-05	1.1112e-05
0.1	6	1.40564	8.99901e-06	6.40204e-06
0.1	7	1.42889	5.72848e-06	4.00901e-06
0.1	8	1.44644	3.8632e-06	2.67083e-06
0.1	9	1.46014	2.72494e-06	1.86622e-06
0.1	10	1.47113	1.99223e-06	1.35422e-06
0.001	1	0.785393	5.10138e-06	6.49528e-06
0.001	2	1.10715	1.63677e-06	1.47837e-06
0.001	3	1.24905	6.1082e-07	4.89029e-07
0.001	4	1.32582	2.8078e-07	2.11779e-07
0.001	5	1.3734	1.49668e-07	1.08976e-07
0.001	6	1.40565	8.85257e-08	6.29786e-08
0.001	7	1.4289	5.64804e-08	3.95272e-08
0.001	8	1.44644	3.81555e-08	2.63789e-08
0.001	9	1.46014	2.69499e-08	1.84571e-08
0.001	10	1.47113	1.97249e-08	1.3408e-08

Ratios of Relative and Absolute Errors

h1	h2	x	abs-err-ratio	rel-err-ratio
0.5	0.1	1	0.184326	0.184326
0.5	0.1	2	0.169199	0.169199

§8.4#7

0.5	0.1	3	0.175251	0.175251
0.5	0.1	4	0.180465	0.180465
0.5	0.1	5	0.184069	0.184069
0.5	0.1	6	0.186611	0.186611
0.5	0.1	7	0.188477	0.188477
0.5	0.1	8	0.189898	0.189898
0.5	0.1	9	0.191012	0.191012
0.5	0.1	10	0.191908	0.191908
0.1	0.001	1	0.00969793	0.00969793
0.1	0.001	2	0.00962195	0.00962195
0.1	0.001	3	0.00970268	0.00970268
0.1	0.001	4	0.00976445	0.00976445
0.1	0.001	5	0.00980704	0.00980704
0.1	0.001	6	0.00983727	0.00983727
0.1	0.001	7	0.00985959	0.00985959
0.1	0.001	8	0.00987665	0.00987665
0.1	0.001	9	0.00989009	0.00989009
0.1	0.001	10	0.00990092	0.00990092

Trapezoidal Method (lambda=-50)
Relative and Absolute Errors

h	x	yh	error	rel-error
0.5	1	0.7865	-0.00110187	-0.00140294
0.5	2	1.10785	-0.000704112	-0.000635969
0.5	3	1.24953	-0.000480137	-0.000384403
0.5	4	1.32616	-0.000340824	-0.000257067
0.5	5	1.37365	-0.000245073	-0.000178442
0.5	6	1.40582	-0.000177057	-0.000125961
0.5	7	1.42903	-0.000128179	-8.97048e-05
0.5	8	1.44653	-9.28869e-05	-6.42176e-05
0.5	9	1.46021	-6.73488e-05	-4.61249e-05
0.5	10	1.47118	-4.88482e-05	-3.32046e-05
0.1	1	0.785407	-8.37744e-06	-1.06665e-05
0.1	2	1.10715	-3.00682e-06	-2.71582e-06
0.1	3	1.24905	-8.85055e-07	-7.08585e-07
0.1	4	1.32582	-3.24428e-07	-2.447e-07
0.1	5	1.3734	-1.42383e-07	-1.03672e-07
0.1	6	1.40565	-7.1288e-08	-5.07154e-08
0.1	7	1.4289	-3.93542e-08	-2.75416e-08
0.1	8	1.44644	-2.34048e-08	-1.61809e-08
0.1	9	1.46014	-1.47555e-08	-1.01056e-08
0.1	10	1.47113	-9.74851e-09	-6.62655e-09
0.001	1	0.785398	-8.31122e-10	-1.05822e-09
0.001	2	1.10715	-3.01225e-10	-2.72073e-10
0.001	3	1.24905	-8.86395e-11	-7.09658e-11
0.001	4	1.32582	-3.24769e-11	-2.44958e-11
0.001	5	1.3734	-1.42522e-11	-1.03773e-11
0.001	6	1.40565	-7.13452e-12	-5.07561e-12
0.001	7	1.4289	-3.93818e-12	-2.7561e-12
0.001	8	1.44644	-2.33857e-12	-1.61678e-12
0.001	9	1.46014	-1.47504e-12	-1.01021e-12

§**8.4#7**

```
0.001    10      1.47113 -9.71445e-13  -6.6034e-13

Ratios of Relative and Absolute Errors
   h1    h2    x    abs-err-ratio    rel-err-ratio
--------------------------------------------------
  0.5   0.1    1       0.00760296       0.00760296
  0.5   0.1    2       0.00427037       0.00427037
  0.5   0.1    3       0.00184334       0.00184334
  0.5   0.1    4      0.000951893      0.000951893
  0.5   0.1    5      0.000580981      0.000580981
  0.5   0.1    6      0.000402626      0.000402626
  0.5   0.1    7      0.000307025      0.000307025
  0.5   0.1    8       0.00025197       0.00025197
  0.5   0.1    9      0.000219091      0.000219091
  0.5   0.1   10      0.000199568      0.000199568
--------------------------------------------------
  0.1 0.001    1      9.92095e-05      9.92095e-05
  0.1 0.001    2      0.000100181      0.000100181
  0.1 0.001    3      0.000100151      0.000100151
  0.1 0.001    4      0.000100105      0.000100105
  0.1 0.001    5      0.000100098      0.000100098
  0.1 0.001    6       0.00010008       0.00010008
  0.1 0.001    7       0.00010007       0.00010007
  0.1 0.001    8      9.99187e-05      9.99187e-05
  0.1 0.001    9      9.99653e-05      9.99653e-05
  0.1 0.001   10      9.96507e-05      9.96507e-05
```

The implicit methods were stable for all values of λ and h. When $\lambda = -1$ the Euler explicit method was stable for $h = 0.5, 0.1$ and 0.001; when $\lambda = -10$ the Euler explicit method was stable only for $h = 0.1$ and 0.001; when $\lambda = -50$ the Euler explicit method was stable only for $h = 0.001$. Note that the trapezoidal method performed as 2nd order as indicated by its greater accuracy and the fact that the errors decreased by a factor of about 1/10000 when the step size decreased by 1/100. Thus, the error tolerances used in the solver for the implicit methods are suitable for preserving the order of the respective methods.

§8.5#2b

Compute solutions to

$$\begin{cases} y' = \dfrac{1}{1+x^2} - 2y^2, & 0 \le x \le 10 \\ y(0) = 0 \end{cases}$$

using a second-order Taylor's method with stepsizes of $h = 0.2$, 0.1 and 0.05. Compute the error and relative error using the true answer $y(x) = x/(1+x^2)$. Compare your results with those in Problem 1 Section 8.2.

We create the Taylor integrator `taylor85p2b` using Maple and include the machine optimized calculations directly into the routime. The argument `f` in `taylor85p2b` is ignored but included so the calling syntax is still compatible with the `testit` function given in the first question. The Maple script to create the Taylor's method is

```
restart;
with(codegen):
f:=1/(1+x^2)-2*y(x)^2;
s:={diff(y(x),x)=f};
df:=subs(s,diff(f,x));
s2:=[y(x)=yn,x=xn];
tayl:=subs(s2,y(x)+h*f+h^2/2*df);
taylo:=optimize(['yn'=tayl]);
C([taylo]);
```

the Taylor's method is

```
function [x,y]=taylor85p2b(x0,y0,xn,f,N)
  h=(xn-x0)/N;
  xn=x0;
  yn=y0;
  for n=1:N

% WARNING: The argument f is not used; instead we directly
% insert the following block generated by Maple
      t1 = xn*xn;
      t2 = 1.0+t1;
      t4 = yn*yn;
      t6 = 1/t2-2.0*t4;
      t8 = h*h;
      t9 = t2*t2;
      yn = yn+h*t6+t8*(-2.0/t9*xn-4.0*yn*t6)/2.0;

    xn=x0+h*n;
    x(:,n)=xn;
    y(:,n)=yn;
  end;
```

which is called by the script

§8.5#2b

```
clear all
f=@(x,y) 1/(1+x^2)-2*y^2;
fexact=@(x) x./(1+x.^2);
x0=0; y0=0; xt=10;
hn=[0.2,0.1,0.05];
testit(fexact,@taylor85p2b,x0,y0,xt,f,hn);
```

and gives the output

```
Relative and Absolute Errors
    h     x          yh        error    rel-error
-------------------------------------------------
  0.2     1    0.498664   0.00133589   0.00267178
  0.2     2    0.398177   0.00182265   0.00455663
  0.2     3    0.299525  0.000475279   0.00158426
  0.2     4    0.235195  9.90941e-05   0.00042115
  0.2     5    0.192307  2.79063e-07  1.45113e-06
  0.2     6    0.162186  -2.4132e-05 -0.000148814
  0.2     7    0.140028  -2.7552e-05   -0.0001968
  0.2     8    0.123102 -2.51525e-05 -0.000204364
  0.2     9    0.109778 -2.15089e-05  -0.00019597
  0.2    10   0.0990279 -1.79997e-05 -0.000181797
-------------------------------------------------
  0.1     1    0.499656  0.000344432  0.000688864
  0.1     2    0.399616  0.000383682  0.000959205
  0.1     3    0.299905  9.49783e-05  0.000316594
  0.1     4    0.235277   1.6992e-05   7.2216e-05
  0.1     5     0.19231 -2.72199e-06 -1.41543e-05
  0.1     6    0.162169 -7.11066e-06 -4.38491e-05
  0.1     7    0.140007 -7.31635e-06 -5.22596e-05
  0.1     8    0.123083 -6.45181e-06  -5.2421e-05
  0.1     9    0.109762 -5.42797e-06 -4.94549e-05
  0.1    10   0.0990144 -4.50178e-06 -4.54679e-05
-------------------------------------------------
 0.05     1    0.499913  8.72919e-05  0.000174584
 0.05     2    0.399911  8.88633e-05  0.000222158
 0.05     3    0.299978  2.16165e-05  7.20549e-05
 0.05     4    0.235291  3.61151e-06  1.53489e-05
 0.05     5    0.192309 -8.83194e-07 -4.59261e-06
 0.05     6    0.162164 -1.84147e-06 -1.13557e-05
 0.05     7    0.140002 -1.84496e-06 -1.31783e-05
 0.05     8    0.123079 -1.61184e-06 -1.30962e-05
 0.05     9    0.109757 -1.35021e-06 -1.23019e-05
 0.05    10    0.099011 -1.11729e-06 -1.12846e-05

Ratios of Relative and Absolute Errors
   h1    h2    x    abs-err-ratio    rel-err-ratio
-------------------------------------------------
  0.2   0.1    1         0.257829         0.257829
  0.2   0.1    2         0.210507         0.210507
  0.2   0.1    3         0.199837         0.199837
  0.2   0.1    4         0.171473         0.171473
  0.2   0.1    5         -9.75401         -9.75401
  0.2   0.1    6         0.294657         0.294657
```

§**8.5#2b**

0.2	0.1	7	0.265547	0.265547
0.2	0.1	8	0.256507	0.256507
0.2	0.1	9	0.252359	0.252359
0.2	0.1	10	0.250103	0.250103
0.1	0.05	1	0.253437	0.253437
0.1	0.05	2	0.231607	0.231607
0.1	0.05	3	0.227594	0.227594
0.1	0.05	4	0.212542	0.212542
0.1	0.05	5	0.324466	0.324466
0.1	0.05	6	0.258973	0.258973
0.1	0.05	7	0.252169	0.252169
0.1	0.05	8	0.249828	0.249828
0.1	0.05	9	0.248751	0.248751
0.1	0.05	10	0.248188	0.248188

Observe that the errors decrease by a quarter when h decreases by half. Thus the approximations given by the Taylor's method are more accurate than given by Euler's method.