Math/CS 467/667 Programming Project 2

## Approximation of Derivatives by FFT

**1.** Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function with period 2. Approximate $f$ on the interval $[-1, 1]$ as $f \approx A$ where

$$A(x) = \sum_{j=-N/2+1}^{N/2} y_j e^{i\pi jx} \quad \text{and} \quad y_j = \frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} f\Big(\frac{2\ell}{N}\Big) e^{-2\pi i\ell j/N}.$$

Differentiate $A$ to obtain approximations for $f'$, $f''$ and $f'''$.

**2.** Modify the approximation in part 1 by setting $y_{N/2} = 0$ to obtain $\widetilde{A}$. Explain why $\widetilde{A}$ is guaranteed to be real for all values of $x$.

**3.** Let $f(x) = \exp(\sin \pi x)$. Compute $f'$, $f''$ and $f'''$ exactly.

**4.** Define

$$A'_\ell = A'\Big(\frac{2\ell}{N}\Big), \qquad A''_\ell = A''\Big(\frac{2\ell}{N}\Big), \qquad A'''_\ell = A'''\Big(\frac{2\ell}{N}\Big),$$
$$\widetilde{A}'_\ell = \widetilde{A}'\Big(\frac{2\ell}{N}\Big), \qquad \widetilde{A}''_\ell = \widetilde{A}''\Big(\frac{2\ell}{N}\Big), \qquad \widetilde{A}'''_\ell = \widetilde{A}'''\Big(\frac{2\ell}{N}\Big).$$

Write a program that uses the the FFT and inverse FFT to compute these approximations for $N = 4, 8, 16$. Display your results in a table form. Are the imaginary parts of $A'_\ell$, $A''_\ell$ and $A'''_\ell$ zero? How about the imaginary parts of $\widetilde{A}'_\ell$, $\widetilde{A}''_\ell$ and $\widetilde{A}'''_\ell$? Which are better approximations? What role does rounding error play?

**5.** Compute the errors

$$E_k = \bigg( \frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} \Big| A_\ell^{(k)} - f^{(k)}\Big(\frac{2\ell}{N}\Big) \Big|^2 \bigg)^{1/2}$$

and

$$\widetilde{E}_k = \bigg( \frac{1}{N} \sum_{\ell=-N/2+1}^{N/2} \Big| \widetilde{A}_\ell^{(k)} - f^{(k)}\Big(\frac{2\ell}{N}\Big) \Big|^2 \bigg)^{1/2}$$

for $k = 1, 2, 3$ and $N = 4, 8, \ldots, 65536$. Comment on the quality of the approximations.

**6.** [Extra Credit] Repeat parts 3 through 5 above for the function $f(x) = \exp(x^2)$. Compare the size of the errors and rate of convergence as $N \to \infty$ in this case to the previous one. Explain any differences or similarities.