Split Runge-Kutta Schemes

Consider a differential equation of the form

$$\frac{dy}{dt} + ay = f(y,t) \qquad \text{with} \qquad y(t_n) = y_n.$$

Introduce the variable $w = e^{a(t-t_n)}y$. Since

$$\frac{dw}{dt} = e^{a(t-t_n)}\frac{dy}{dt} + e^{a(t-t_n)}ay = e^{a(t-t_n)}f(y,t)$$

then

$$\frac{dw}{dt} = g(w,t) \qquad \text{where} \qquad g(w,t) = e^{a(t-t_n)}f(we^{-a(t-t_n)},t).$$

Integrate using the Euler method

$$w_{n+1} = w_n + h\,g(w_n,t_n)$$

Rewriting the above in terms of y yields

$$y_{n+1} = e^{-ah}(y_n + h\,f(y_n,t_n))$$

Integrate using the two-stage second-order Runge-Kuttla method

$$\begin{aligned} k_1 &= h\,g(w_n,t_n) \\ k_2 &= h\,g(w_n+k_1,t_n+h) \\ w_{n+1} &= w_n + (k_1+k_2)/2 \end{aligned}$$

Rewriting the above in terms of y yields

$$\begin{aligned} k_1 &= h\,f(y_n,t_n) \\ k_2 &= h\,e^{ah}f((y_n+k_1)e^{-ah},t_n+h) \\ y_{n+1} &= e^{-ah}(y_n + (k_1+k_2)/2) \end{aligned}$$

Exercise 1. Rewrite the four-stage fourth-order Runge-Kutta scheme

$$\begin{aligned} k_1 &= h\,g(w_n,t_n) \\ k_2 &= h\,g(w_n+k_1/2,t_n+h/2) \\ k_3 &= h\,g(w_n+k_2/2,t_n+h/2) \\ k_4 &= h\,g(w_n+k_3,t_n+h) \\ w_{n+1} &= w_n + (k_1+2k_2+2k_3+k_4)/6 \end{aligned}$$

in terms of y.