Math/CS 467/667 Practice Final Exam Version A

This exam consists of four questions. The first two questions are closed book, closed computer and closed notes; the second two questions are open book, open computer and open notes. All questions have parts which should be completed using pencil and paper. Some parts of the third and fourth questions should be completed using the computer and submitted electronically. Please place all work to be submitted electronically in the subdirectory `final` and then use the command `submit final` to submit it. Please ask for help if you have any trouble with the electronic submission or if you have any other questions.

1. Prove the following theorem on the convergence of Euler's method:

Theorem. Consider the ordinary differential equation initial-value problem

$$\frac{dy}{dt} = f(t,y) \qquad \text{with} \qquad y(t_0) = y_0$$

where $|f_y(t,\xi)| \le B$ for $\xi \in \mathbf{R}$ and $t \in [t_0, T]$. Suppose there exists a unique solution y such that $|y''(t)| \le A$ for $t \in [t_0, T]$. Then Euler's method for approximating y given by

$$y_{k+1} = h_k + hf(t_k, y_k) \qquad \text{where} \qquad t_k = t_0 + kh \quad \text{and} \quad h = \frac{T - t_0}{n}$$

satisfies the limit $\big|y_n - y(T)\big| \to 0$ as $n \to \infty$.

2. Define

$$L_h f = -\frac{15}{4h^3}\int_{-h}^{h}\Big(1-\frac{3t^2}{h^2}\Big)f(t)dt$$

(i) Use Taylor's theorem to show that $L_h f = f''(0) + \mathcal{O}(h^2)$.

(ii) Suppose f_ε is a function such that

$$\big|f(t) - f_\varepsilon(t)\big| \le \varepsilon \qquad \text{for every} \qquad t \in \mathbf{R}.$$

Show that $\big|L_h f_\varepsilon - f''(0)\big| \le 15\varepsilon/h^2 + \mathcal{O}(h^2)$.

3. Consider the integral

$$\int_a^b f(x)\,dx \qquad \text{where} \qquad a = 0, \quad b = 5 \quad \text{and} \quad f(x) = x\sin x.$$

Further consider Simpson's formula

$$S(\alpha, \beta, f) = \frac{\beta - \alpha}{6}\Big(f(\alpha) + 4f\Big(\frac{\alpha + \beta}{2}\Big) + f(\beta)\Big)$$

and the resulting quadrature method given by

$$Q_N(a, b, f) = \sum_{j=0}^{N-1} S(x_j, x_{j+1}, f)$$

where $N \in \mathbf{N}$ and $x_j = a + hj$ with $h = (b - a)/N$.

(i) Use the rules of Calculus to find the exact value A of the integral.

(ii) Write a program to compute $E_N = \big|Q_N(a, b, f) - A\big|$ for $N = 2^p$ where $p = 4, 5, \ldots 16$. Place your program and its output in the subdirectory `final` and use `submit final` to electronically submit your answer.

(iii) [Extra Credit] Plot E_N versus N using log-log scale to verify the order of Simpson's quadrature method numerically. Submit your plot and the script used to make it in the same `final` subdirectory as used above.

4. Consider the solutions to the equation

$$f(z) = 0 \qquad \text{where} \qquad f(z) = 2\sin z + z^2 + 1$$

and the application of Newton's method

$$z_{n+1} = g(z_n) \qquad \text{where} \qquad g(z) = z - f(z)/f'(z)$$

and z_0 is an initial guess to find the roots.

(i) Write a program to compute the resulting limits for initial guesses of the form $z_0 = a + ib$ where $(a, b) \in [0, 4] \times [0, 4]$. Plot the basin of attraction for Newton's method by coloring each point in $[0, 4] \times [0, 4]$ based on the limit. Place your program and its output in the subdirectory `final` and use `submit final` to electronically submit your answer.

(ii) Interpret the color of each point in your plot by explaining what it means. In particular, what limit value does each color correspond to?