Math/CS 467/667 Homework 1 Solutions Part 2

14.6a Suppose $f(t)$ is a smooth function and $y_k = f(kh)$ for $k \in \{-n, -n+1, \ldots, 0, \ldots n\}$. Show that the parabola $p(t) = at^2 + bt + c$ optimally fitting these data points via least squares satisfies

$$p'(0) = \sum_k k y_k \Big/ \sum_k h k^2.$$

For notational convenience write $t_k = kh$. Least squares finds a, b and c which minimize the function

$$\Phi(a,b,c) = \sum_k \big(p(t_k) - y_k\big)^2.$$

Since the index set for k is symmetric about the origin then

$$\sum_k t_k^j = 0 \qquad \text{for every} \qquad j \text{ odd.}$$

At the minimum, setting the partial derivative equal to zero yields

$$\begin{aligned}\partial_b \Phi(a,b,c) &= 2\sum_k \big(p(t_k) - y_k\big)t_k = 2\sum_k \big(at_k^3 + bt_k^2 + ct_k\big) - 2\sum_k y_k t_k \\ &= 2b\sum_k t_k^2 - 2\sum_k y_k t_k = 2b\sum_k h^2k^2 - 2\sum_k y_k hk = 0.\end{aligned}$$

Therefore

$$b = \sum_k y_k hk \Big/ \sum_k h^2 k^2 = \sum_k k y_k \Big/ \sum_k hk^2$$

as required.

14.6b Use this formula to propose approximations of $f'(0)$ when $n = 1, 2, 3$.

When $n = 1$ approximate

$$f'(0) \approx p'(0) = \sum_{k=-1}^{1} k y_k \Big/ \sum_{k=-1}^{1} hk^2 = \frac{f(h) - f(-h)}{2h}.$$

When $n = 2$ approximate

$$f'(0) \approx p'(0) = \sum_{k=-2}^{2} k y_k \Big/ \sum_{k=-2}^{2} hk^2 = \frac{2f(2h) + f(h) - f(-h) - 2f(-2h)}{10h}.$$

When $n = 3$ approximate

$$f'(0) \approx p'(0) = \frac{3f(3h) + 2f(2h) + f(h) - f(-h) - 2f(-2h) - 3f(-3h)}{28h}.$$

14.6c Motivate the following formula for differentiation by integration:

$$f'(0) = \lim_{h\to 0} \frac{3}{2h^3} \int_{-h}^{h} tf(t)dt.$$

By Taylor's theorem

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(\xi)$$

for some ξ between 0 and t. Consequently, since the integral of an odd power of t over a symmetric interval is zero, we obtain

$$\begin{aligned} \int_{-h}^{h} tf(t)dt &= \int_{-h}^{h} \Big(tf(0) + t^2 f'(0) + \frac{t^3}{2} f''(0) + \frac{t^4}{6} f'''(\xi) \Big) dt \\ &= \int_{-h}^{h} \Big(t^2 f'(0) + \frac{t^4}{6} f'''(\xi) \Big) dt = \frac{2h^3}{3} f'(0) + \int_{-h}^{h} \frac{t^4}{6} f'''(\xi) dt \end{aligned}$$

Since f is smooth than f''' is continuous on $[-1,1]$. Let M be the bound such that

$$\big|f'''(t)\big| \le M \qquad \text{for} \qquad t \in [-1,1].$$

It follows that

$$\Big| \frac{3}{2h^3} tf(t)dt - f'(0) \Big| = \Big| \frac{3}{2h^3} \int_{-h}^{h} \frac{t^4}{6} f'''(\xi) dt \Big| \le \frac{3}{2h^3} \frac{2h^5}{30} M \to 0 \quad \text{as} \quad h \to 0.$$

14.6d Show that when $h > 0$ that

$$\frac{3}{2h^3} \int_{-h}^{h} tf(t)dt = f'(0) + \mathcal{O}(h^2).$$

This is exactly what I showed in the previous problem in order to motivate the differentiation by integration formula.

14.6e Denote

$$D_h f = \frac{3}{2h^3} \int_{-h}^{h} tf(t)dt.$$

Suppose, thanks to noise, we actually observe $f^\varepsilon(t)$ satisfying $\big|f(t) - f^\varepsilon(t)\big| \le \varepsilon$ for all t. Show that

$$\big|D_h f^\varepsilon - f'(0)\big| \le \frac{3\varepsilon}{2h} + \mathcal{O}(h^2).$$

Estimate

$$\begin{aligned}\big|D_h f^\varepsilon - D_h f\big| &= \Big|\frac{3}{2h^3}\int_{-h}^{h} t\big(f^\varepsilon(t) - f(t)\big)\,dt\Big| \\ &\le \frac{3}{2h^3}\int_{-h}^{h} |t|\varepsilon\,dt = \frac{3}{2h^3}\frac{2h^2}{2}\varepsilon = \frac{3\varepsilon}{2h}.\end{aligned}$$

Consequently, by the triangle inequality

$$\big|D_h f^\varepsilon - f'(0)\big| \le \big|D_h f^\varepsilon - D_h f\big| + \big|D_h f - f'(0)\big| \le \frac{3\varepsilon}{2h} + \mathcal{O}(h^2).$$

14.6f Suppose the second term in the previous part is bounded above by $Mh^2/10$ as is the case when $\big|f'''(t)\big| \le M$ everywhere. Show that with the right choice of h, the integral approximation is within $\mathcal{O}(\varepsilon^{2/3})$ of $f'(0)$.

Take $h = \varepsilon^{1/3}$. Then

$$\big|D_h f^\varepsilon - f'(0)\big| \le \frac{3\varepsilon}{2h} + \frac{Mh^2}{10} = \frac{3\varepsilon}{2\varepsilon^{1/3}} + \frac{M\varepsilon^{2/3}}{10} = \mathcal{O}(\varepsilon^{2/3}).$$